VIBRATIONS OF ANISOTROPIC SHALLOW SHELLS

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DEDICATED TO MY MOTHER

CERTIFICATE

Certified that this work on 'Vibrations of Shallow Anisotropic Shells' has been carried out under my supervision and that this has not been submitted elsewhere for a degree.

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POST GRADUATE OFFICE
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CONTENTS

LIST OF SYMBOLS

A, B	: Lame parameters
e_1, e_2, e_n	: Unit vectors
e_α, e_β	: Longitudinal strains
$e_{\alpha\beta}, e_\beta, e_\alpha$	: Shear strains
A_{ik}, B_{ik}	: Elastic constants
E	: Young's Modulus
g	: Acceleration due to gravity
h	: Shell thickness
K_1, K_2	: Curvatures
M_1, M_2, H	: Moments
R_1, R_2	: Principal radii of curvatures
T_1, T_2, S	: Membrane forces
u, v, w	: Displacements of a point on the middle surface
α, β, γ	: Triorthogonal coordinates on middle surface
γ^*	: Density of the material
ν	: Poisson's ratio
ω	: Circular Frequency
ω^*	: Non-dimensionalised frequency ratio

ABSTRACT

This dissertation is an analytical study of free vibration characteristics or the elastic response of thin shallow anisotropic doubly curved shells used as roofs and ceilings in industrial and domestic buildings.

A short review of the available literature in the field of vibrations of shells is presented. The free vibration problem is to determine an infinitesimal displacement field which satisfies the equations of motion and boundary conditions in the absence of any body or surface forces. For strain-displacement relations the hypothesis of Kirchoff-Love is assumed. Further the Lame parameters are assumed to be constants in differentiation. For stress-strain relations a plane of elastic symmetry parallel to the middle surface of the shell is assumed. Hamiltonian principle is used to get variational equation of motion. Variation of displacement gives equilibrium equations and boundary conditions. After getting an eighth order partial differential equation, Kantorovich method is used to separate the variables. The resulting ordinary differential equation with boundary conditions forms linear eigen value problem. For non-trivial

solution, the determinant is made zero or it is minimised by an iterative technique. Closed form solution by double sine series for all sides simply supported is obtained to check the results. Variation of frequency ratio with different shell parameters and boundary conditions is presented.

CHAPTER I

INTRODUCTION

Thin walled shells, operating mainly under compression, pos ess great strength and stability for comparatively low weight, which is thus the logical reason for their use in engineering construction. The present work is devoted to investigations of free vibration characteristics or the elastic response of thin shallow doubly curved shells used primarily as roofs and ceilings in industrial and domestic buildings.

Special literature devoted to questions dynamic problems of elastic shells is small. A. Kalnins |1|* has stated the mathematical formulation of the boundary value problem of vibrations of shells in various stages of approximation and has reviewed the methods of finding free-vibration and transient solutions to this problem.

1.1 MATHEMATICAL MODELS FOR VIBRATION OF SHELLS

In order to avoid some mathematical difficulties in the analysis of vibration of shells, many simplifications in the governing equations have been employed. For detailed

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* The integer numbers within the bracket refer to bibliography

discussion of these approximations, let us assume an elastic shell whose points are described by the curvilinear co-ordinates α and β, lying in a given reference surface, and γ, which is the distance from the reference surface measured along its normal. The components of the displacement vector are designated by $u_i(\alpha, \beta, \gamma, t)$.

The free vibration problem is to determine an infinitesimal displacement field u_i which satisfies the approximate equations of motion and boundary conditions in the absence of any body or surface forces. Assuming that the most general free-vibration analysis of an elastic system is given by the equations of the three dimensional linear theory of elasticity, the particular approximations of the displacement field used in the analysis of elastic shells can be listed in the following way:

(a) Normals to the reference surface remain straight lines during deformations; or

$$u_i(\alpha, \beta, \gamma, t) = u_i^o(\alpha, \beta, t) + \gamma\, \phi_i(\alpha, \beta, t)$$

(b) Normals to the reference surface do not extend during deformation; or

$$e_{33} = u_{3,3} = 0$$

where e_{33} is the transverse normal strain component.

(c) Normals to the reference surface remain normal during deformation; or

$$2e_{i3} = u_{i,3} + u_{3,i} = 0$$

(d) Rotary inertia about tangents of the α co-ordinate curves is neglected; or

$$I_i = 0$$

where I_i is the mass moment of inertia per unit volume.

(e) Tangential translatory inertia is neglected; or

$$\rho_i = 0$$

where ρ_i is the mass density of the material in the direction of the tangent of the α_i coordinate curve.

(f) Bending stiffness of shell is zero.

(g) Extensional stiffness of shell is infinite.

Making one or more of these approximations, many types of theories can be constructed which have been employed in past for the purpose of analysing the free-vibration of shell.

1.2 IMPROVED THEORY OF SHELLS

The approximation (a) can be regarded as a basic one which distinguishes a 'shell theory' from the theory of elasticity, and it leads directly from the governing equations of the theory of elasticity to a general shell theory as shown by Reissner |2| and Naghdi |3|. The specific effects of assumptions (a) and (b) can be determined by comparing the free vibration spectra as given by the theory of elasticity and the improved shell theory. Such comparisons for cylindrical shells are made by Gazis |4| and Greenspon |5|.

As far as free vibration of shells is concerned, approximations (a) and (b) will affect the frequency spectrum in the range where the thickness modes begin to appear.

With the use of improved theory of shells, the vibration of only the simplest shell shapes has been considered: cylindrical shells of infinite length by Lin and Morgan |6| and Naghdi |7|, shallow spherical shells by Kalnins |8|, Non-shallow spherical shells by Prasad |9|, Wilkinson |10|, conical shells by Garnet and Kempner |11|. In refences |6-10|, exact, separable, homogeneous solutions are employed from which frequency equation can be constructed

from boundary conditions. Thus any natural frequency can be obtained exactly within the scope of the theory. The Rayleigh-Ritz or Galerkin type analysis of |11| is applied to obtain the upper bound of the lowest mode of free vibration of truncated cones.

1.3 CLASSICAL (BENDING) THEORY OF SHELLS

Assumptions (c) and (d), in addition to (a) and (b) lead to the classical shell theory, which was first formulated by Love |12| in 1888. This paper contains a complete system of equations of the bending theory of shells which, with a few changes, is still in use today. These changes in Love's equations are necessary mainly because the twist term given in |12| is not symmetric with respect to the two coordinates of the reference of the shell. This is corrected by Reissner |13| and later by Naghdi |14|.

The effect of assumptions (c) and (d) is simply that the thickness-shear motion which is still present in the improved theory, is eliminated. Thus the frequency spectrum of the classical theory does not contain any coupling of the thickness modes.

As in the case of improved theory, separable homogeneous solutions, exact within the scope of the classical

bending theory have been derived for cylindrical shells by Arnold and Warburton |15| and for spherical shells by Reissner |16|. Examples of application of such solutions to free vibration problems can be found in many papers |17-20|. Numerical techniques are used by Goldberg and Bogdanoff |21|, Kalnins |22|. This class of shells includes composite shells whose wall material is isotropic, layered or homogeneous.

1.4 THEORY OF TRANSVERSE VIBRATIONS OF SHALLOW SHELLS

If in the governing equations of shallow shells the tangential (or longitudinal) inertia terms are omitted, then solution of the problem can be obtained by means of an Airy-type stress function as shown Reissner |23|. The approximation (e) has been employed by Reissner |16| and Johnson |19| for shallow spherical shells

The effect of approximation (f) can be understood best by recalling that transverse and in plane (or longitudinal) vibration of flat plate is completely uncoupled. For a shallow shell, which can be regarded as just a slightly curved plate, transverse and longitudinal modes do not appear separately but are coupled, and coupling is proportional to curvature. Thus frequency equation derived from exact equation of the classical bending theory of

shallow shells predicts both transverse and longitudinal modes.

The effect of assumption (e) on vibration of shallow shells is simply that those modes which correspond to the longitudinal modes in a plate are eliminated from frequency spectrum. Furthermore Kalnins |24| has shown that the effect of longitudinal inertia on the transverse modes is almost negligible for shallow spherical shells. He has further concluded in |25| that assumption (e) is applicable to only shallow shells and not, in general to non-shallow shells.

1.5 EXTENSIONAL (MEMBRANE) VIBRATIONS

Another type of vibration of shells, which was explored by Lamb |26| and Love |12|, is derived from the assumption that the shell has no bending stiffness i.e. the strain energy of deformation is produced by stretching of the reference surface only. They have investigated this type of vibration in open and closed spherical shells and both concluded that the spectrum of the modes of free vibration consist of two distinct types: (1) a set of an infinite number of modes spaced within a finite frequency interval at a lower end of the spectrum and (2) another set of infinite number of modes which are above those of the first set.

It is shown by Kalnins in |25| that assumption (f) may affect considerably the 'bending' modes of the frequency spectrum but not the 'membrane' modes. For a closed spherical shell these two sets of modes are very distinct; however, for more complicated shell shapes this may not be so, for it may be very difficult to tell which of the two modes predicted by the membrane theory are in error and which ones are accurate.

1.6 INEXTENSIONAL VIBRATION

The concept of 'inextensional' deformation employed in the theory of flat plates, has been proposed by Lord Rayleigh |27| for theory of shells. According to Rayleigh's theory, the three displacement components of a particular shell are obtained from three differential equations which result when the three middle surface strain-displacement relations are set equal to zero. Such a displacement field has been derived by Rayleigh |27| for general shell of revolution of second degree, which can be used to predict the upper bound of one natural frequency by means of Rayleigh's principle. Because only bending energy appears in the strain-energy, the predicted frequencies are proportional to the thickness of the shell.

1.7 THEORY OF ANISOTROPIC SHELLS

Ambartsumian |28| has presented the existing work

done in the general theory of anisotropic shells. Approximations (a), (b) and (c) i.e. the hypothesis of non-deformable normals is assumed and the case when at each point of the shell there is only one plane of elastic symmetry parallel to the middle surface of the shell is considered. The general theory of anisotropic shells is developed in |29-31|.

In the general theory of anisotropic shells the equations of equilibrium, geometric relations, equations of compatibility and boundary conditions remain unchanged in general formulation, i.e. they agree with the corresponding equations of the isotropic shell theory. Anisotropy appears only in the elasticity relations, principally differing from those of isotropic shell theory.

The equations of static stability and vibrations of shallow anisotropic shells in general case of anisotropy are derived in |29|, |30|. The question of determining natural vibration frequencies is discussed by Sakharov |32|, Kutnikova |33| and Oniashvili |34|, and natural frequencies of cylindrical, spherical and conical shells are determined by a conventional classical method.

1.8 METHODS OF SOLUTION OF FREE VIBRATION PROBLEM

The basic governing equations of vibration of shells consists of 13 differential equations (5 equilibrium and 8 strain-displacement equations) plus 8 algebraic stress

resultant-strain equations. Any method of analysis of shells must start with the reduction of these 21 equations to a manageable system of equations. Two such methods have been successfully employed for shells of revolution: (A) reduction of the 21 equations to a single N^{th} order differential equation involving a single unknown; and (B) reduction to an equivalent system of N first order differential equations involving N unknowns. The number N depends upon the type of shell theory used.

In the method (A) the usual procedure is to eliminate from the system of equations all the dependent variables except one, which is usually taken as a transverse displacement of the reference surface denoted by W. The resulting single differential equation may be written in the form

$$\left[A_{N/2} \, \square^{N} + \cdots + A_2 \, \square^{4} + A_1 \, \square^{2} + A_0 \right] W = 0 \tag{1.1}$$

where $\square^2$ denotes a second-order differential operator and the A_i are constants (with respect to coordinates α and β) which contain geometric and elastic parameters of shells and derivatives with respect to time. Upon separation of variables, (1.1)* becomes an ordinary N^{th} order differential equation

* The number in the bracket refers to equation. First digit is Chapter number.

with constant coefficient A_i, whose general solution consists of a sum of the solutions of

$$\left[\Box^2 - \lambda_i\right] W = 0 \tag{1.2}$$

λ_i are N/2 roots of the algebraic equation

$$A_{N/2} \lambda^{N/2} + \ldots + A_2 \lambda^2 + A_1 \lambda + A_0 = 0 \tag{1.3}$$

The general solution of the homogeneous boundary value problem can be written for every independent variable y_i as

$$y_i(\alpha) = \sum_{i=1}^{N} C_{ij} B_i W_i(\alpha) \tag{1.4}$$

where C_{ij} are constant factors and B_i are N arbitrary constants. With the use of (1.4), the frequency equation is easily constructed from N homogeneous boundary conditions which must be given with the statement of the problem.

The solutions $W_i(\alpha)$ for cylindrical and spherical shells are given by the usual hypergeometric expansions; in particular, $W_i(\alpha)$ for a cylindrical shell are trigonometric functions, for a shallow spherical shell they are Bessel functions, while for a non-shallow spherical shell they are Legendre functions.

Method (B) is based on the idea that the boundary value problem of a shell of revolution can be stated in the form of a system of N first order differential equations, containing N unknowns, subject to N boundary conditions. The one and the only restriction of this method is that the shell (but not the vibration) must be rotationally symmetric i.e. all geometric and elastic properties of shell can vary arbitrarily (even discontinuously) along the meridian of the reference surface of the shell but not along its circumference.

The boundary value problem is stated in terms of exactly N unknowns denoted by $y_i(\alpha)$ in the form

$$\frac{dy_i(\alpha)}{dx} = F_i(\alpha, y_1, y_2, \ldots y_N) \qquad (1.5)$$

and N boundary conditions at two values of α. This boundary value problem is then replaced by N initial value problems to which solution can be obtained by means of either hypergeometric series (for cylindrical and spherical shells) or direct numerical integration (for arbitrary rotationally symmetric shells).

Out of these methods, in the present work method (A) is extended to derive natural frequencies and mode shapes of doubly curved anisotropic shallow shells. Hamiltonian principle

is used to get variational equation of motion. Variation of displacement gives equilibrium equations and proper boundary conditions.

After getting an eighth order coupled differential equation, Kantorovich method is used to separate the variables. As an illustrative example the method is degenerated to Levy type solution. The roots of the eighth order characteristic equation are found by using the scheme introduced by N.S.V.K. Rao |35|.

Vibration problem with boundary conditions forms linear eigen value problem. For non-trivial solution, the determinant is made zero. An iterative technique is used to minimise the residue.

In this work the derivation of the equations is given in second chapter. Third chapter explains the Kantorovich method. An illustrative example by Levy type solution is presented in fourth chapter. Numerical results and conclusions are drawn in fifth chapter.

The flow chart of the computer programme written for the numerical procedure is given in appendix. The programme is written in Fortran IV language for IBM 7044.

CHAPTER II

DERIVATION OF GOVERNING EQUATIONS

In this chapter the governing differential equations are derived on the basis of small deformation theory A thin-walled, sufficiently shallow and anisotropic shell is considered, whose material, at each point, has a plane of elastic symmetry parallel to the middle surface of the shell.

Following assumptions are taken from |30|:

(1) The hypothesis of Kirchoff-Love shows that the rectilinear elements of the shell normal to the middle surface maintain their initial length after deformation of shell. The error of this hypothesis, as shown by V.V. Novozilov, has a value of the order of (hk) compared with unity, where h is the constant thickness of the shell and k is Gaussian curvature.

(2) The Lame parameters are regarded as constants in differentiation.

(3) Certain terms of secondary significance are neglected.

Hamiltonian principle is used to deduce variational equation of motion. Variation of displacements gives equilibrium equations and proper boundary conditions. Assuming Donnel-Vlasov type stress-function the problem is reduced to the form of a two-equation system with respect to a displacement function and a function of stresses, and finally, to the form of an eighth order equation with respect to one potential function.

2.1 SOME RESULTS FROM THEORY OF SURFACES

An element of the surface is shown in Figure (2.1). Two principal lines of curvatures α and β are chosen as coordinate lines. The lines α and β are called Gaussian curvilinear coordinates. On the surface an elemental length ds is given by the following relation:

$$ds^2 = A^2\, d\alpha^2 + B^2 d\beta^2 \tag{2.1}$$

where A and B are Lame's parameters. The principal radii of curvatures corresponding to α and β are denoted by R_1 and R_2 respectively. The Lame parameters and radii of curvatures are functions of α and β.

The unit vectors along α and β are $\hat{e}_1$ and $\hat{e}_2$ and are shown in Figure (2.1). They are obtained by following expressions:

$$\hat{e}_1 = \frac{1}{A}\frac{\partial \bar{r}}{\partial \alpha}$$

$$\hat{e}_2 = \frac{1}{B}\frac{\partial \bar{r}}{\partial \beta} \qquad (2.2)$$

where, $\bar{r}\,(\alpha, \beta)$ is position vector of a point on the surface. A third direction, γ , shown in the Figure (2.1) is chosen such that a unit vector along that direction is

$$\hat{e}_n = \hat{e}_1 \times \hat{e}_2$$

The three vectors $\hat{e}_1$, $\hat{e}_2$ and $\hat{e}_n$ for a right hand triad.

The Lame parameters A and B and curvature radii R_1 and R_2 satisfy the well known Gauss-Codazzi relations, which are

$$\frac{\partial}{\partial \alpha}\left(\frac{B}{R_2}\right) = \frac{1}{R_1}\frac{\partial B}{\partial \alpha}$$

$$\frac{\partial}{\partial \beta}\left(\frac{A}{R_1}\right) = \frac{1}{R_2}\frac{\partial A}{\partial \beta} \qquad (2.3)$$

$$\frac{\partial}{\partial \alpha}\left(\frac{1}{A}\frac{\partial B}{\partial \alpha}\right) + \frac{1}{\partial \beta}\left(\frac{1}{B}\frac{\partial A}{\partial \beta}\right) = -\frac{AB}{R_1 R_2}$$

2.2 DEFORMATION OF SHELL

The deformed state of shell is determined by the components, u, v, w of the displacement vector, whose positive directions are shown in Figure (2.2).

The hypothesis of Kirchoff-Love used in deriving strain-displacement relations. These are derived in |36| for thin shallow shells.

$$e_\alpha = \varepsilon_1 + \gamma\, k_1$$

$$e_\beta = \varepsilon_2 + \gamma\, k_2 \qquad (2.4)$$

$$e_{\alpha\beta} = \varepsilon_{12} + \gamma\, k_{12}$$

$$e_{\alpha\gamma} = e_{\beta\gamma} = e_\gamma = 0$$

where

$$\varepsilon_1 = \frac{1}{A}\frac{\partial u}{\partial \alpha} + \frac{1}{AB}\frac{\partial A}{\partial \beta} v + \frac{w}{R_1}$$

$$(2.4a)$$

$$\varepsilon_2 = \frac{1}{B}\frac{\partial v}{\partial \beta} + \frac{1}{AB}\frac{\partial B}{\partial \alpha} u + \frac{w}{R_2}$$

$$\varepsilon_{12} = \frac{A}{B}\frac{\partial}{\partial \beta}\left(\frac{u}{A}\right) + \frac{B}{A}\frac{\partial}{\partial \alpha}\left(\frac{v}{B}\right)$$

$$k_1 = \frac{1}{A}\frac{\partial}{\partial \alpha}\left(\frac{1}{A}\frac{\partial w}{\partial \alpha}\right) - \frac{1}{AB^2}\frac{\partial A}{\partial \beta}\frac{\partial w}{\partial \beta}$$

$$k_2 = \frac{1}{B}\frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial w}{\partial\beta}\right) - \frac{1}{A^2B}\frac{\partial B}{\partial\alpha}\frac{\partial w}{\partial\alpha}$$

$$k_{12} = -\frac{2}{AB}\left(\frac{\partial^2 w}{\partial\alpha\partial\beta} - \frac{1}{A}\frac{\partial A}{\partial B}\frac{\partial w}{\partial\alpha} - \frac{1}{B}\frac{\partial B}{\partial\alpha}\frac{\partial w}{\partial\beta}\right) \qquad (2.4b)$$

Here e_1 and e_2 are the relative elongations in the direction of the coordinate lines α and β respectively; e_{12} is the relative shear which is characterised by a change in the angle between the coordinate lines α = constant, β = constant; k_1 and k_2 are the components of the deformation related to a change in the curvature of the middle surface in the directions of coordinate lines α and β respectively; and k_{12} is the relative twisting deformation. Being shallow shells the effect of inplane displacements u and v on k_1, k_2 and k_{12} is neglected.

Ambertsumian |30| further assumed the Lame parameters A and B as constants in differentiation. In the light of this assumption, equations (2.4a)and (2.4b) become

$$e_1 = \frac{1}{A}\frac{\partial u}{\partial\alpha} + \frac{w}{R_1}$$

$$e_2 = \frac{1}{B}\frac{\partial v}{\partial\beta} + \frac{w}{R_2} \qquad (2.5a)$$

$$e_{12} = \frac{1}{B}\frac{\partial u}{\partial\beta} + \frac{1}{A}\frac{\partial v}{\partial\alpha}$$

$$k_1 = -\frac{1}{A^2}\frac{\partial^2 w}{\partial \alpha^2}$$

$$k_2 = -\frac{1}{B^2}\frac{\partial^2 w}{\partial \beta^2}$$

$$k_{12} = -\frac{2}{AB}\frac{\partial^2 w}{\partial \alpha \partial \beta} \qquad (2.5b)$$

2.3 RELATIONS BETWEEN DEFORMATIONS AND STRESSES

The thin walled, sufficiently shallow and anisotropic shell is assumed to have, at each point, a plane of elastic symmetry parallel to the middle surface of the shell. The equations of the generalised Hooke's law in the chosen triorthogonal curvilinear system of coordinates are

$$\sigma_\alpha = A_{11}e_\alpha + A_{12}e_\beta + A_{13}e + A_{16}e_{\alpha\beta}$$

$$\sigma_\beta = A_{12}e_\alpha + A_{22}e_\beta + A_{23}e + A_{26}e_{\alpha\beta}$$

$$\sigma_\gamma = A_{13}e_\alpha + A_{23}e_\beta + A_{33}e + A_{36}e_{\alpha\beta}$$

$$\tau_{\beta\gamma} = A_{44}e_\beta + A_{45}e_\alpha$$

$$\tau_{\alpha\gamma} = A_{45}e_\beta + A_{55}e_\alpha$$

$$\tau_{\alpha\beta} = A_{16}e_\alpha + A_{26}e_\beta + A_{36}e + A_{66}e_{\alpha\beta} \qquad (2.6)$$

In the case under consideration for $\sigma_\gamma = 0$ we have

$$\sigma_\alpha = B_{11}e_\alpha + B_{12}e_\beta + B_{16}e_{\alpha\beta}$$

$$\sigma_\beta = B_{12}e_\alpha + B_{22}e_\beta + B_{26}e_{\alpha\beta}$$

$$\tau_{\alpha\beta} = B_{16}e_\alpha + B_{26}e_\beta + B_{66}e_{\alpha\beta} \qquad (2.7)$$

where, following S.G. Lekhnitskii's theories |37|, there is introduced the notation

$$B_{ik} = (A_{ik}A_{33} - A_{i3}A_{k3})/A_{33}$$

$$i, k = 1,2,6$$

Figure (2.3) shows the element of a shell and stresses on the surface. The face is denoted according to the direction of the normal on it.

Denoting T_1 as force acting on the face α in the direction α, S as force acting on the face α in the direction β and N_1 as force acting on the face α in γ direction. All forces are per unit length of α. Similarly forces on β are denoted.

In the same way moments are defined. Let M_1 and H be the moments acting on the face α. Moments are taken positive when the moment vector is acting on the positive

face in the positive direction. Positive face is defined when normal is directed along positive direction. Forces on the faces α and β in terms of stress resultants are

$$T_1 = \int_{-h/2}^{h/2} \sigma_\alpha d\gamma$$

$$= h(B_{11}e_1 + B_{12}e_2 + B_{16}e_{12})$$

$$T_2 = h(B_{12}e_1 + B_{22}e_2 + B_{26}e_{12})$$

$$S_1 = -S_2 = S = h(B_{16}e_1 + B_{26}e_2 + B_{66}e_{12}) \tag{2.8}$$

$$M_1 = -\int_{-h/2}^{h/2} \sigma_\alpha \gamma \, d\gamma$$

$$= -\frac{h^3}{12}(B_{11}k_1 + B_{12}k_2 + B_{16}k_{12})$$

$$M_2 = -\frac{h^3}{12}(B_{12}k_1 + B_{22}k_2 + B_{26}k_{12})$$

$$H_1 = -H_2 = H = \frac{h^3}{12}(B_{16}k_1 + B_{26}k_2 + B_{66}k_{12}) \tag{2.8a}$$

2.4 EQUILIBRIUM EQUATIONS

Equations of motion are deduced from the Hamiltonian

principle [38]. Let T be the total kinetic energy of the body, and V to be the potential energy of deformation, so that V is the volume integral of strain-energy-function W. Then by rules of calculus of variations, the variation of the integral $\int(T-V)dt$ is taken between fixed initial and final values (t_0 and t_1) for t. In varying the integral it is assumed that the displacement alone is subject to variation, and that its values at the initial and final instants are given. The variation so formed is $\delta\int(T-V)dt$. Let δW_1 be the work done by the external forces when displacement is varied. Then the principle is expressed by the equation:

$$\delta\int(T-V)dt + \int \delta W_1 \, dt = 0 \tag{2.9}$$

we have

$$T = \iiint \frac{1}{2} \rho \left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2 \right] dxdydz$$

$$\delta \int T \, dt = \int dt \iiint \rho \left(\frac{\partial u}{\partial t}\frac{\partial \delta u}{\partial t} + \dots + \dots\right) dxdydz$$

$$= \int_{t_0}^{t_1} \iiint \rho \left(\frac{\partial u}{\partial t}\delta u + \frac{\partial v}{\partial t}\delta v + \frac{\partial w}{\partial t}\delta w\right) dxdydz$$

$$- \int dt \iiint \rho \left(\frac{\partial^2 u}{\partial t^2}\delta u + \frac{\partial^2 v}{\partial t^2}\delta v + \frac{\partial^2 w}{\partial t^2}\delta w\right) dxdydz \tag{2.9a}$$

Here t_0 and t_1 are initial and final values of t, and δu, δv, δw vanish for both these values. The first term may therefore be omitted, and the equation (2.9) is then transformed into a variational equation of motion. Further

$$\delta V = \iiint \delta W \, dxdydz$$

and

$$\delta W_1 = 0 \tag{2.9b}$$

Hence variational equation of motion is of the form

$$\iiint \left[\delta W + \rho \left(\frac{\partial^2 u}{\partial t^2} \delta u + \frac{\partial^2 v}{\partial t^2} \delta v + \frac{\partial^2 w}{\partial t^2} \delta w \right) \right] dxdydz = 0 \tag{2.10}$$

For the case under consideration

$$W = \frac{1}{2} (\sigma_\alpha e_\alpha + \sigma_\beta e_\beta + \tau_{\alpha\beta} e_{\alpha\beta})$$

$$dxdydz = ABd\alpha d\beta d\gamma$$

Hence (2.10) becomes

$$\iiint \left[\delta (\sigma_\alpha e_\alpha + \sigma_\beta e_\beta + \tau_{\alpha\beta} e_{\alpha\beta}) \frac{1}{2} \right.$$

$$\left. + \left(\frac{\partial^2 u}{\partial t^2} \delta u + \frac{\partial^2 v}{\partial t^2} \delta v + \frac{\partial^2 w}{\partial t^2} \delta w \right) \right] ABd\alpha d\beta d\gamma = 0 \tag{2.10a}$$

Now substituting values of ${}_{\alpha}$, ${}_{\beta}$, ${}_{\alpha\beta}$ from (2.7) in the first term of the above we get

$$\frac{1}{2}\iiint \delta\left[(B_{11}e_\alpha + B_{12}e_\beta + B_{16}e_{\alpha\beta})e_\alpha\right.$$

$$+ (B_{12}e_\alpha + B_{22}e_\beta + B_{26}e_{\alpha\beta})e_\beta$$

$$\left.+ (B_{16}e_\alpha + B_{26}e_\beta + B_{66}e_{\alpha\beta})e_{\alpha\beta}\right] ABd\alpha d\beta d\gamma$$

Varying e_α, e_β and $e_{\alpha\beta}$ in the above and integrating with respect to γ, we get from (2.8) following form of the variational equation (2.10a)

$$\iint(T_1\delta\mathcal{E}_1 + T_2\delta\mathcal{E}_2 + S\delta\mathcal{E}_{12})\ ABd\alpha d\beta$$

$$- \iint(M_1\delta k_1 + M_2\delta k_2 - H\delta k_{12})\ ABd\alpha d\beta$$

$$- \rho h\iint(\frac{\partial^2 u}{\partial t^2}\delta u + \frac{\partial^2 v}{\partial t^2}\delta v + \frac{\partial^2 w}{\partial t^2}\delta w)ABd\alpha d\beta = 0 \qquad (2.11)$$

Simplifying further the first term of the above becomes

$$\iint T_1\delta\mathcal{E}_1 ABd\alpha d\beta$$

$$= \iint T_1(\frac{1}{A}\frac{\partial\delta u}{\partial\alpha} + \frac{\delta w}{R_1})\ ABd\alpha d\beta$$

$$= \int\left[BT_1\delta u\right]_0^{\alpha_0} d\beta - \iint\frac{\partial}{\partial\alpha}(BT_1)\delta u d\alpha d\beta$$

$$+ \iint T_1\delta w ABd\alpha d\beta$$

Similarly proceeding with the remaining terms we can equate to zero the coefficients of δu, δv and δw, which will be three equilibrium equations and remaining will give proper boundary conditions.

$$\delta u: \quad B\frac{\partial T_1}{\partial \alpha} + A\frac{\partial S}{\partial \beta} + \rho h \frac{\partial^2 u}{\partial t^2} AB = 0$$

$$\delta v: \quad A\frac{\partial T_2}{\partial \beta} + B\frac{\partial S}{\partial \alpha} + \rho h \frac{\partial^2 v}{\partial t^2} AB = 0$$

$$\delta w: \quad \frac{T_1}{R_1} + \frac{T_2}{R_2} + \frac{h^3}{12} L\,(a_{ik})w + \rho h \frac{\partial^2 w}{\partial t^2} = 0 \qquad (2.12a)$$

Neglecting the derivatives of δw we can get the following boundary conditions.

Along $\alpha = 0$ and $\alpha = \alpha_o$

$$\int \left[BT_1\delta u + BS\delta v + \left(\frac{B}{A}\frac{\partial M_1}{\partial \alpha} + \frac{\partial H}{\partial \beta}\right) \delta w \right] d\beta$$

Along $\beta = 0$ and $\beta = \beta_o$

$$\int \left[AT_2\delta v + AS\delta u + \left(\frac{A}{B}\frac{\partial M_2}{\partial \beta} + \frac{\partial H}{\partial \alpha}\right) \delta w \right] d\alpha \qquad (2.12b)$$

2.5 EQUATIONS OF COMPATIBILITY

There are three equations of compatibility derived by A.L. Goldenweiser |39| which connect e_{11}, e_{22}, e_{12}, k_1, k_2 and k_{12}. These are:

$$\frac{\partial}{\partial\beta}(Ak_1) - k_2\frac{\partial A}{\partial\beta} - \frac{\partial(Bk_{12})}{\partial\alpha} - k_{12}\frac{\partial B}{\partial\alpha} + \frac{\varepsilon_{12}}{R_1}\frac{\partial B}{\partial\alpha}$$

$$- \frac{1}{R_2}\left(\frac{\partial(A\varepsilon_1)}{\partial\beta} - \frac{\partial(B\varepsilon_{12})}{\partial\alpha} - \varepsilon_2\frac{\partial A}{\partial\beta}\right) = 0$$

(2.13)

$$\frac{\partial}{\partial\alpha}(Bk_2) - k_1\frac{\partial B}{\partial\alpha} - \frac{\partial(Ak_{12})}{\partial\beta} - k_{12}\frac{\partial A}{\partial\beta} - \frac{\partial A}{\partial\beta}\frac{\varepsilon_{12}}{R_2}$$

$$- \frac{1}{R_1}\left(\frac{\partial(B\varepsilon_2)}{\partial\alpha} - \frac{\partial(A\varepsilon_{12})}{\partial\beta} - \varepsilon_1\frac{\partial B}{\partial\alpha}\right) = 0$$

$$\frac{k_1}{R_2} + \frac{k_2}{R_1} + \frac{1}{AB}\Bigg[\frac{\partial}{\partial\alpha}\frac{1}{A}\Bigg\{A_2\frac{\partial\varepsilon_2}{\partial\alpha} + \frac{\partial B}{\partial\alpha}(\varepsilon_2 - \varepsilon_1)$$

$$- \frac{1}{2}A\frac{\partial\varepsilon_{12}}{\partial\beta} - \frac{\partial A}{\partial\beta}\varepsilon_{12}\Bigg\} + \frac{\partial}{\partial\beta}\frac{1}{B}\Bigg\{A\frac{\partial\varepsilon_1}{\partial\beta}$$

$$+ \frac{\partial A}{\partial\beta}(\varepsilon_1 - \varepsilon_2) - \frac{1}{2}B\frac{\partial\varepsilon_{12}}{\partial\alpha} - \frac{\partial B}{\partial\alpha}\varepsilon_{12}\Bigg\}\Bigg] = 0$$

Of these, for the case under consideration, only the last one is important.

2.6 FUNDAMENTAL DIFFERENTIAL EQUATIONS

For solution of (2.12) Donnel and Vlasov |40| type functions are used. They are

$$T_1 = \frac{1}{B^2}\frac{\partial^2\varphi}{\partial\beta^2}$$

$$T_2 = \frac{1}{A^2} \frac{\partial^2 \varphi}{\partial \alpha^2}$$

$$S = -\frac{1}{AB} \frac{\partial^2 \varphi}{\partial \alpha \partial \beta} \qquad (2.14)$$

With the help of this stress function, the first two of equilibrium equations (2.12) are identically satisfied provided we set

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 v}{\partial t^2} = 0 \qquad (2.15)$$

The third of (2.12) becomes

$$\nabla_r \varphi + \frac{h^3}{12} L\,(B_{ik})\, w + \rho h \frac{\partial^2 w}{\partial t^2} = 0 \qquad (2.16)$$

When (2.5a) and (2.5b) are substituted in the third of (2.13) we get

$$-\nabla_r w + \frac{1}{\delta \Omega} L_1\,(B_{ik})\, \varphi = 0 \qquad (2.17)$$

where

$$\nabla_r = \frac{1}{R_1} \frac{1}{B^2} \frac{\partial^2}{\partial \beta^2} + \frac{1}{R_2} \frac{1}{A^2} \frac{\partial^2}{\partial \alpha^2} \qquad (2.18)$$

$$L_1(B_{ik}) = C_1 \frac{\partial^4}{\partial \alpha^4} + C_2 \frac{\partial^4}{\partial \alpha^3 \partial \beta} + C_3 \frac{\partial^4}{\partial \alpha^2 \partial \beta^2} +$$

$$+ C_4 \frac{\partial^4}{\partial \alpha \partial \beta^3} + C_5 \frac{\partial^4}{\partial \beta^4} \qquad (2.19a)$$

where

$$c_1 = \frac{1}{B_{66}} (B_{11}B_{66} - B_{16}^2) \frac{1}{A^4}$$

$$c_2 = \frac{2}{B_{66}} (B_{11}B_{26} - B_{12}B_{16}) \frac{1}{A^3B}$$

$$c_3 = \frac{1}{B_{66}} \left[(B_{11}B_{22} - B_{12}^2) - 2(B_{12}B_{66} - B_{16}B_{26})\right] \frac{1}{A^2B^2}$$

$$c_4 = \frac{2}{B_{66}} (B_{22}B_{16} - B_{12}B_{26}) \frac{1}{AB^3}$$

$$c_5 = \frac{1}{B_{66}} (B_{22}B_{66} - B_{26}^2) \frac{1}{B^4} \qquad (2.19b)$$

$$L(B_{ik}) = d_1 \frac{\partial^4}{\partial\alpha^4} + d_2 \frac{\partial^4}{\partial\alpha^3\partial\beta} + d_3 \frac{\partial^4}{\partial\alpha^2\partial\beta^2}$$

$$+ d_4 \frac{\partial^4}{\partial\alpha\partial\beta^3} + d_5 \frac{\partial^4}{\partial\beta^4} \qquad (2.20a)$$

where

$$d_1 = B_{11} \frac{1}{A^4}$$

$$d_2 = 4B_{16} \frac{1}{A^3B}$$

$$d_3 = 2(B_{12} + 2B_{66}) \frac{1}{A^2B^2}$$

$$d_4 = 4B_{26} \frac{1}{AB^3}$$

$$d_5 = B_{22} \frac{1}{B^4} \tag{2.20b}$$

$$= \frac{1}{B_{66}^2}\left[(B_{11}B_{66} - B_{16}^2)(B_{22}B_{66} - B_{26}^2) - (B_{12}B_{66} - B_{16}B_{26})^2\right] \tag{2.21}$$

From (2.16) and (2.17), for $R_1 = R_2 = \infty$, we obtain the well known equations for the plane stress state of a plate $L_1(B_{ik}) = 0$, and for the vibration of an anisotropic plate,

$$L(B_{ik})w + \frac{12\rho}{h^2}\frac{\partial^2 w}{\partial t^2} = 0$$

Thus, making use of the mixed method of V.Z. Vlasov, we obtain a more compact representation of the differential equations of the theory of anisotropic shells. The system (2.16) and (2.17) may be reduced to an equivalent single equation of the eighth order.

The equation (2.16) is identically satisfied by letting

$$w = L_1(B_{ik})\,\phi$$

$$\varphi = \delta\Omega\,\nabla_r\,\phi \tag{2.22}$$

Then (2.17) becomes

$$L_1(B_{ik})\, L(B_{ik})\, \phi + \frac{12\,\Omega}{h^2}\, \nabla_r^2\, \phi$$

$$+ \frac{12\,\rho}{h^2}\, \frac{\partial^2}{\partial t^2}\, L_1(B_{ik})\, \phi = 0 \qquad (2.23)$$

The internal forces, by (2.5a), (2.5b), (2.8) and (2.22), are as follows:

$$T_1 = h\,\Omega\, \frac{1}{B^2}\, \frac{\partial^2}{\partial \beta^2}\, \nabla_r\, \phi$$

$$T_2 = h\,\Omega\, \frac{1}{A^2}\, \frac{\partial^2}{\partial \beta^2}\, \nabla_r\, \phi$$

$$S = -h\,\Omega\, \frac{1}{AB}\, \frac{\partial^2}{\partial \alpha \partial \beta}\, \nabla_r\, \phi \qquad (2.24)$$

$$M_1 = \frac{h^3}{12}\Big[B_{11}\, \frac{1}{A^2}\, \frac{\partial^2}{\partial \alpha^2} + B_{12}\, \frac{1}{B^2}\frac{\partial^2}{\partial \beta^2}$$

$$+ B_{16}\, \frac{2}{AB}\, \frac{\partial^2}{\partial \alpha \partial \beta}\Big]\, L_1(B_{ik})\, \phi$$

$$M_2 = \frac{h^3}{12}\Big[B_{12}\, \frac{1}{A^2}\, \frac{\partial^2}{\partial \alpha^2} + B_{22}\, \frac{1}{B^2}\, \frac{\partial^2}{\partial \beta^2}$$

$$+ B_{26}\, \frac{2}{AB}\, \frac{\partial^2}{\partial \alpha \partial \beta}\Big]\, L_1(B_{ik})\, \phi$$

$$H = -\frac{h^3}{12}\left[B_{16}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2} + B_{26}\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}\right.$$

$$\left. + B_{66}\frac{2}{AB}\frac{\partial^2}{\partial\beta\partial\alpha}\right]L_1(B_{ik})\phi$$

$$N_1 = -\frac{h^3}{12}C(B_{ik})L_1(B_{ik})\phi \qquad (2.26)$$

$$N_2 = -\frac{h^3}{12}D(B_{ik})L_1(B_{ik})\phi$$

where

$$C(B_{ik}) = B_{11}\frac{1}{A^3}\frac{\partial^3}{\partial\alpha^3} + 3B_{16}\frac{1}{A^2B}\frac{\partial^3}{\partial\alpha^2\partial\beta}$$

$$+ (B_{12} + 2B_{66})\frac{1}{AB^2}\frac{\partial^3}{\partial\alpha\partial\beta^2} + B_{26}\frac{1}{B^3}\frac{\partial^3}{\partial\beta^3}$$

$$(2.27)$$

$$D(B_{ik}) = B_{22}\frac{1}{B^3}\frac{\partial^3}{\partial\beta^3} + 3B_{26}\frac{1}{AB^2}\frac{\partial^3}{\partial\alpha\partial\beta^2}$$

$$+ (B_{12} + 2B_{66})\frac{1}{A^2B}\frac{\partial^3}{\partial\beta\partial\alpha^2} + B_{16}\frac{1}{A^3}\frac{\partial^3}{\partial\alpha^3}$$

For the displacement of the middle surface, we have

$$w = L_1(B_{ik})\phi \qquad (2.28)$$

$$u = e_1\frac{\partial^3\phi}{\partial\alpha^3} + e_2\frac{\partial^3\phi}{\partial\alpha^2\partial\beta} + e_3\frac{\partial^3\phi}{\partial\alpha\partial\beta^2} + e_4\frac{\partial^3\phi}{\partial\beta^3} \qquad (2.29a)$$

where

$$e_1 = -\frac{1}{B_{66}}\left[(B_{11}B_{66} - B_{16}^2)\frac{1}{R_1} + (B_{12}B_{66} - B_{16}B_{26})\frac{1}{R_2}\right]\frac{1}{A^3}$$

$$e_2 = -\frac{1}{B_{66}}\left[2(B_{11}B_{26} - B_{12}B_{16})\frac{1}{R_1} - (B_{22}B_{16} - B_{12}B_{26})\frac{1}{R_2}\right]\frac{1}{A^2B}$$

$$e_3 = -\frac{1}{B_{66}}\left[\left\{(B_{11}B_{22} - B_{12}^2) - (B_{12}B_{26} - B_{16}B_{26})\right\}\frac{1}{R_1} - (B_{22}B_{66} - B_{26}^2)\frac{1}{R_2}\right]\frac{1}{AB^2}$$

$$e_4 = -\frac{1}{B_{66}}(B_{22}B_{16} - B_{12}B_{26})\frac{1}{R_1}\frac{1}{B^3} \tag{2.29b}$$

$$v = f_1\frac{\partial^3\phi}{\partial\alpha^3} + f_2\frac{\partial^3\phi}{\partial\alpha^2\partial\beta} + f_3\frac{\partial^3\phi}{\partial\alpha\partial\beta^2} + f_4\frac{\partial^3\phi}{\partial\beta^3} \tag{2.30a}$$

where

$$f_1 = -\frac{1}{B_{66}}(B_{11}B_{26} - B_{12}B_{16})\frac{1}{R_2}\frac{1}{A^3}$$

$$f_2 = -\frac{1}{B_{66}}\left[\left\{(B_{11}B_{22} - B_{12}^2) - (B_{12}B_{66} - B_{16}B_{26})\right\}\frac{1}{R_2} - (B_{11}B_{66} - B_{16}^2)\frac{1}{R_1}\right]\frac{1}{A^2B}$$

$$f_3 = -\frac{1}{B_{66}}\Big[2(B_{22}B_{16} - B_{12}B_{26})\frac{1}{R_2}$$

$$- (B_{11}B_{26} - B_{12}B_{16})\frac{1}{R_1}\Big]\frac{1}{AB^2}$$

$$f_4 = -\frac{1}{B_{66}}\Big[(B_{22}B_{66} - B_{26}^2)\frac{1}{R_2}$$

$$+ (B_{12}B_{66} - B_{16}B_{26})\frac{1}{R_1}\Big]\frac{1}{B^3}$$

(2.30b)

CHAPTER III

NATURAL FREQUENCIES AND MODE SHAPES OF DOUBLY CURVED SHELLS - KANTOROVICH METHOD

In the previous chapter the governing eighth-order differential equation is derived. The present chapter deals with the general solution of the same by Kantorovich method to obtain natural frequencies and mode shapes of doubly curved shells.

3.1 KANTOROVICH METHOD

Equation (2.23) can be written in the form

$$F(\phi) = 0 \tag{3.1a}$$

where

$$F = \frac{h^2}{12} L_1(B_{1k})L(B_{1k}) + \Omega \nabla_r^2 + \rho \frac{\partial^2}{\partial t^2} L_1(B_{1k}) \tag{3.1b}$$

From (2.19) and (2.20) we have

$$\begin{aligned} L_1(B_{1k})L(B_{1k}) = {} & a_1 \frac{\partial^8}{\partial \alpha^8} + a_2 \frac{\partial^8}{\partial \alpha^7 \partial \beta} + a_3 \frac{\partial^8}{\partial \alpha^6 \partial \beta^2} \\ & + a_4 \frac{\partial^8}{\partial \alpha^5 \partial \beta^3} + a_5 \frac{\partial^8}{\partial \alpha^4 \partial \beta^4} + a_6 \frac{\partial^8}{\partial \alpha^3 \partial \beta^5} \\ & + a_7 \frac{\partial^8}{\partial \alpha^2 \partial \beta^6} + a_8 \frac{\partial^8}{\partial \alpha \partial \beta^7} + a_9 \frac{\partial^8}{\partial \beta^8} \end{aligned} \tag{3.2a}$$

where

$$a_1 = e_1 d_1$$

$$a_2 = e_1 d_2 + e_2 d_1$$

$$a_3 = e_1 d_3 + e_2 d_2 + e_3 d_1$$

$$a_4 = e_1 d_4 + e_2 d_3 + e_3 d_2 + e_4 d_1$$

$$a_5 = e_1 d_5 + e_2 d_4 + e_3 d_3 + e_4 d_2 + e_5 d_1$$

$$a_6 = e_2 d_5 + e_3 d_4 + e_4 d_3 + e_5 d_2$$

$$a_7 = e_3 d_5 + e_4 d_4 + e_5 d_3$$

$$a_8 = e_4 d_5 + e_5 d_4$$

$$a_9 = e_5 d_5 \tag{3.2b}$$

The variational equation of (3.1) becomes

$$\iint F(\phi)\, L_1(B_{1k})\delta\phi \; d\alpha d\beta = 0 \tag{3.3}$$

After substitution of $L_1(B_{1k})$ from (2.19), the integration is carried out. The first term yields the following:

$$c_1 \int_0^{\beta_o} \left[F(\phi) \frac{\partial^3 \delta\phi}{\partial\alpha^3} - \frac{\partial F(\phi)}{\partial\alpha} \frac{\partial^2 \delta\phi}{\partial\alpha^2} + \frac{\partial^2 F(\phi)}{\partial\alpha^2} \frac{\partial \delta\phi}{\partial\alpha} \right.$$

$$\left. - \frac{\partial^3 F(\phi)}{\partial\alpha^3} \delta\phi \right]_0^{\alpha_o} d\beta +$$

$$+ \iint c_1 \frac{\partial^4 F(\phi)}{\partial\alpha^4} \delta\phi \; d\alpha d\beta$$

Proceeding on the same lines for the remaining terms and neglecting the derivatives of $\delta\phi$ with respect to α and β, the following equation is obtained:

$$- \int \left[c_1 \frac{\partial^3 F(\phi)}{\partial\alpha^3} + c_2 \frac{\partial^3 F(\phi)}{\partial\alpha^2 \partial\beta} + \frac{c_3}{2} \frac{\partial^3 F(\phi)}{\partial\alpha \partial\beta^2} \right]_0^{\alpha_o} \delta\phi \; d\beta$$

$$- \int \left[c_5 \frac{\partial^3 F(\phi)}{\partial\beta^2} + c_4 \frac{\partial^3 F(\phi)}{\partial\beta^2 \partial\alpha} + \frac{c_3}{2} \frac{\partial^3 F(\phi)}{\partial\beta \partial\alpha^2} \right]_0^{\beta_o} \delta\phi \; d\alpha$$

$$+ \iint L_1(B_{ik}) \; F(\phi) \delta\phi \; d\alpha d\beta \quad = \quad 0 \qquad (3.4)$$

The first two terms vanish on the respective boundaries and hence can be neglected. The last term can be written as:

$$L_1(B_{ik}) \left[\iint F(\phi) \; \delta\phi \; d\alpha d\beta \right] \quad = \quad 0 \qquad (3.5a)$$

Or simply as

$$\iint F(\phi) \; \delta\phi \; d\alpha d\beta \quad = \quad 0 \qquad (3.5b)$$

Let

$$\phi(\alpha,\beta,t) = \sum_{m=1}^{M} X_m(\alpha)\, Y_m(\beta) \sin \omega t \qquad (3.6)$$

where X_m and Y_m are purely functions of α and β respectively. ω is the frequency of vibration. In Kantorovich method it is further assumed that $X_m(\alpha)$ is a known function which satisfies all the geometric boundary conditions along $\alpha = 0$ and $\alpha = \alpha_o$. Hence

$$\delta\phi = X_k(\alpha)\, \delta Y_k(\beta) \sin \omega t \qquad (3.7)$$

Substituting (3.6) and (3.7) in (3.5b) taking into account (3.1b), (3.2a) and (3.2b), the following equation is obtained.

$$\int_0^{\beta_0} \Bigg\{ \int_0^{\alpha_0} \Bigg[\sum_{m=1}^{M} \frac{h^2}{12} \Bigg(a_1 \frac{d^8 X_m}{d\alpha^8} Y_m + a_2 \frac{d^7 X_m}{d\alpha^7} \frac{dY_m}{d\beta}$$

$$+ a_3 \frac{d^6 X_m}{d\alpha^6} \frac{d^2 Y_m}{d\beta^2} + a_4 \frac{d^5 X_m}{d\alpha^5} \frac{d^3 Y_m}{d\beta^3} + a_5 \frac{d^4 X_m}{d\alpha^4} \frac{d Y^4_m}{d\beta^4}$$

$$+ a_6 \frac{d^3 X_m}{d\alpha^3} \frac{d^5 Y_m}{d\beta^5} + a_7 \frac{d^2 X_m}{d\alpha^2} \frac{d^6 Y_m}{d\beta^6} + a_8 \frac{dX_m}{d\alpha} \frac{d^7 Y_m}{d\beta^7}$$

$$+ a_9 \frac{X_m\, d^8 Y_m}{d\beta^8} \Bigg) + \Omega \Bigg(b_1 \frac{d^4 X_m}{d\alpha^4} Y_m + b_2 \frac{d^2 X_m}{d\alpha^2} \frac{d^2 Y_m}{d\beta^2}$$

$$+ b_3 X_m \frac{d^4 Y_m}{d\beta^4} \Bigg) - \frac{\gamma}{g} \omega^2 \Bigg(c_1 \frac{d^4 X_m}{d\alpha^4} Y_m$$

$$+ c_2 \frac{d^3 X_m}{d\alpha^3} \frac{dY_m}{d\beta} + c_3 \frac{d^2 X_m}{d\alpha^2} \frac{d^2 Y_m}{d\beta^2} + c_4 \frac{dX_m}{d\alpha} \frac{d^3 Y_m}{d\beta^3}$$

$$+ c_5 X_m \frac{d^4 Y_m}{d\beta^4} \Bigg) \Bigg] X_k \, d\alpha \Bigg\} \delta Y_k \, d\beta = 0 \qquad (3.8)$$

$$(k = 1, 2, \ldots M)$$

Now if $\delta Y_k \neq 0$, then

$$\sum_{m=1}^{M} \int_{0}^{\alpha_0} \Big[\frac{h^2}{12} a_9 X_m \frac{d^8 Y_m}{d\beta^8} + \frac{h^2}{12} a_8 \frac{dX_m}{d\alpha} \frac{d^7 Y_m}{d\beta^7}$$

$$+ \frac{h^2}{12} a_7 \frac{d^2 X_m}{d\alpha^2} \frac{d^6 Y_m}{d\beta^6} + \frac{h^2}{12} a_6 \frac{d^3 X_m}{d\alpha^3} \frac{d^5 Y_m}{d\beta^5}$$

$$+ \left(\frac{h^2}{12} a_5 \frac{d^4 X_m}{d\alpha^4} + \Omega\, b_3 X_m - \frac{\gamma^*}{g} \omega^2 e_5 X_m\right) \frac{d^4 Y_m}{d\beta^4}$$

$$+ \left(\frac{h^2}{12} a_4 \frac{d^5 X_m}{d\alpha^5} - \frac{\gamma^*}{g} \omega^2 e_4 \frac{dX_m}{d\alpha}\right) \frac{d^3 Y_m}{d\beta^3}$$

$$+ \left(\frac{h^2}{12} a_3 \frac{d^6 X_m}{d\alpha^6} + \Omega\, b_2 \frac{d^2 X_m}{d\alpha^2} - \frac{\gamma^*}{g} \omega^2 e_3 \frac{d^2 X_m}{d\alpha^2}\right) \frac{d^2 Y_m}{d\beta^2}$$

$$+ \left(\frac{h^2}{12} a_2 \frac{d^7 X_m}{d\alpha^7} - \frac{\gamma^*}{g} \omega^2 e_2 \frac{d^3 X_m}{d\alpha^3}\right) \frac{dY_m}{d\beta}$$

$$+ \Big(\frac{h^2}{12} a_1 \frac{d^8 X_m}{d\alpha^8} + \Omega\, b_1 \frac{d^4 X_m}{d\alpha^4}$$

$$- \frac{\gamma^*}{g} \omega^2 e_1 \frac{d^4 X_m}{d\alpha^4}\Big) Y_m \Big] X_k \, d\alpha = 0 \qquad (3.9)$$

$$(k = 1, 2, \ldots M)$$

Now let

$$A_{km} = \int_{0}^{\alpha_0} \frac{h^2}{12} a_9 X_m X_k \, d\alpha$$

$$B_{km} = \int_0^{\alpha_0} \frac{h^2}{12} a_8 \frac{dX_m}{d\alpha} X_k \, d\alpha$$

$$C_{km} = \int_0^{\alpha_0} \frac{h^2}{12} a_7 \frac{d^2X_m}{d\alpha^2} X_k \, d\alpha$$

$$D_{km} = \int_0^{\alpha_0} \frac{h^2}{12} a_6 \frac{d^3X_m}{d\alpha^3} X_k \, d\alpha$$

$$E_{km} = \int_0^{\alpha_0} \left(\frac{h^2}{12} a_5 \frac{d^4X_m}{d\alpha^4} + \Omega \, b_3 X_m - \frac{\gamma^*}{g} \omega^2 e_5 X_m \right) X_k \, d\alpha$$

$$F_{km} = \int_0^{\alpha_0} \left(\frac{h^2}{12} a_4 \frac{d^5X_m}{d\alpha^5} - \frac{\gamma^*}{g} \omega^2 e_4 \frac{dX_m}{d\alpha} \right) X_k \, d\alpha$$

$$G_{km} = \int_0^{\alpha_0} \left(\frac{h^2}{12} a_3 \frac{d^6X_m}{d\alpha^6} + \Omega \, b_2 \frac{d^2X_m}{d\alpha^2} - \frac{\gamma^*}{g} \omega^2 e_3 \frac{d^2X_m}{d\alpha^2} \right) X_k \, d\alpha$$

$$H_{km} = \int_0^{\alpha_0} \left(\frac{h^2}{12} a_2 \frac{d^7X_m}{d\alpha^7} - \frac{\gamma^*}{g} \omega^2 e_2 \frac{d^3X_m}{d\alpha^3} \right) X_k \, d\alpha$$

$$I_{km} = \int_0^{\alpha_0} \left(\frac{h^2}{12} a_1 \frac{d^8 X_m}{d\alpha^8} + \Omega b_1 \frac{d^4 X_m}{d\alpha^4} - \frac{\gamma^*}{g} \omega^2 c_1 \frac{d^4 X_m}{d\alpha^4} \right) X_k \, d\alpha \tag{3.10}$$

Hence (3.9) becomes

$$\sum_{m=1}^{M} \left(A_{km} \frac{d^8 Y_m}{d\beta^8} + B_{km} \frac{d^7 Y_m}{d\beta^7} + C_{km} \frac{d^6 Y_m}{d\beta^6} + \cancel{D_{km} \frac{d^5 Y_m}{d\beta^5}} + E_{km} \frac{d^4 Y_m}{d\beta^4} + F_{km} \frac{d^3 Y_m}{d\beta^3} + G_{km} \frac{d^2 Y_m}{d\beta^2} + H_{km} \frac{d Y_m}{d\beta} + I_{km} Y_m \right) = 0 \tag{3.11}$$

$$(k = 1, 2, \ldots M)$$

Let the solution (3.11) be expressed as

$$Y_m(\beta) = \sum_{m=1}^{M} \sum_{i=1}^{8} C_{im} \, e^{\lambda_{im} \beta} \tag{3.12}$$

Substitution of (3.12) in (3.11) will yield following determinant:

$$\begin{bmatrix} P_{11} & P_{12} & P_{13} & \cdots & P_{1M} \\ P_{21} & P_{22} & P_{23} & \cdots & P_{2M} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ P_{M1} & P_{M2} & P_{M3} & \cdots & P_{MM} \end{bmatrix} = 0 \qquad (3.13a)$$

where

$$P_{km} = A_{km} \lambda_{im}^{8} + B_{km} \lambda_{im}^{7} + C_{km} \lambda_{im}^{6}$$

$$+ D_{km} \lambda_{im}^{5} + E_{km} \lambda_{im}^{4} + F_{km} \lambda_{im}^{3}$$

$$+ G_{km} \lambda_{im}^{2} + H_{km} \lambda_{im} + I_{km} \qquad (3.13b)$$

$$(k, m = 1,2,M)$$
$$(i = 1,2,8)$$

Hence λ and constants C_{im} will have 8M values each.

3.2 RESULTANT FORCES AND DISPLACEMENTS

Substitution of (3.6) in (2.24), (2.25), (2.26) will give internal stresses.

$$T_1 = \sum_{m=1}^{M} \left(d_{T11m} \frac{d^4 Y_m}{d\beta^4} + d_{T12m} \frac{d^2 Y_m}{d\beta^2}\right) \sin \omega t \qquad (3.14a)$$

$$T_2 = \sum_{m=1}^{M} \left(d_{T21m} \frac{d^2 Y_m}{d\beta^2} + d_{T22m} Y_m \right) \sin \omega t \qquad (3.14b)$$

$$S = \sum_{m=1}^{M} \left(d_{S1m} \frac{d^3 Y_m}{d\beta^3} + d_{S2m} \frac{dY_m}{d\beta} \right) \sin \omega t \qquad (3.14c)$$

$$M_1 = \sum_{m=1}^{M} \left(d_{M11m} \frac{d^6 Y_m}{d\beta^6} + d_{M12m} \frac{d^5 Y_m}{d\beta^5} + d_{M13m} \frac{d^4 Y_m}{d\beta^4} \right.$$

$$+ d_{M14m} \frac{d^3 Y_m}{d\beta^3} + d_{M15m} \frac{d^2 Y_m}{d\beta^2} + d_{M16m} \frac{dY_m}{d\beta}$$

$$\left. + d_{M17m} Y_m \right) \sin \omega t \qquad (3.14d)$$

$$M_2 = \sum_{m=1}^{M} \left(d_{M21m} \frac{d^6 Y_m}{d\beta^6} + d_{M22m} \frac{d^5 Y_m}{d\beta^5} + d_{M23m} \frac{d^4 Y_m}{d\beta^4} \right.$$

$$+ d_{M24m} \frac{d^3 Y_m}{d\beta^3} + d_{M25m} \frac{d^2 Y_m}{d\beta^2} + d_{M26m} \frac{dY_m}{d\beta}$$

$$\left. + d_{M27m} Y_m \right) \sin \omega t \qquad (3.14e)$$

$$H = \sum_{m=1}^{M} \left(d_{H1m} \frac{d^6 Y_m}{d\beta^6} + d_{H2m} \frac{d^5 Y_m}{d\beta^5} + d_{H3m} \frac{d^4 Y_m}{d\beta^4} \right.$$

$$+ d_{H4m} \frac{d^3 Y_m}{d\beta^3} + d_{H5m} \frac{d^2 Y_m}{d\beta^2} + d_{H6m} \frac{dY_m}{d\beta}$$

$$\left. + d_{H7m} Y_m \right) \sin \omega t \qquad (3.14f)$$

$$N_1 = \sum_{m=1}^{M} \left(d_{N11m} \frac{d^7 Y_m}{d\beta^7} + d_{N12m} \frac{d^6 Y_m}{d\beta^6} + d_{N13m} \frac{d^5 Y_m}{d\beta^5} \right.$$

$$+ d_{N14m} \frac{d^4 Y_m}{d\beta^4} + d_{N15m} \frac{d^3 Y_m}{d\beta^3} + d_{N16m} \frac{d^2 Y_m}{d\beta^2}$$

$$\left. + d_{N17m} \frac{dY_m}{d\beta} + d_{N18m} Y_m \right) \sin \omega t \qquad (3.14g)$$

$$N_2 = \sum_{m=1}^{M} \left(d_{N21m} \frac{d^7 Y_m}{d\beta^7} + d_{N22m} \frac{d^6 Y_m}{d\beta^6} + d_{N23m} \frac{d^5 Y_m}{d\beta^5} \right.$$

$$+ d_{N24m} \frac{d^4 Y_m}{d\beta^4} + d_{N25m} \frac{d^3 Y_m}{d\beta^3} + d_{N26m} \frac{d^2 Y_m}{d\beta^2}$$

$$\left. + d_{N27m} \frac{dY_m}{d\beta} + d_{N28m} Y_m \right) \sin \omega t \qquad (3.14h)$$

Similarly (2.28), (2.29) and (2.30) will give displacements

$$u = \sum_{m=1}^{M} \left(d_{u1m} \frac{d^3 Y_m}{d\beta^3} + d_{u2m} \frac{d^2 Y_m}{d\beta^2} + d_{u3m} \frac{dY_m}{d\beta} \right.$$

$$\left. + d_{u4m} Y_m \right) \sin \omega t \qquad (3.14i)$$

$$v = \sum_{m=1}^{M} \left(d_{v1m} \frac{d^3 Y_m}{d\beta^3} + d_{v2m} \frac{d^2 Y_m}{d\beta^2} + d_{v3m} \frac{dY_m}{d\beta} \right.$$

$$\left. + d_{v4m} Y_m \right) \sin \omega t \qquad (3.14j)$$

$$w = \sum_{m=1}^{M} \left(d_{w1m} \frac{d^4 Y_m}{d\beta^4} + d_{w2m} \frac{d^3 Y_m}{d\beta^3} + d_{w3m} \frac{d^2 Y_m}{d\beta^2} \right.$$

$$\left. + d_{w4m} \frac{dY_m}{d\beta} + d_{w5m} Y_m \right) \sin \omega t \qquad (3.14k)$$

Slope in β-direction is given by

$$w' = \frac{\partial w}{\partial \beta}$$

$$= \sum_{m=1}^{M} \left(d_{w1m} \frac{d^5 Y_m}{d\beta^5} + d_{w2m} \frac{d^4 Y_m}{d\beta^4} + d_{w3m} \frac{d^3 Y_m}{d\beta^3} \right.$$

$$\left. + d_{w4m} \frac{d^2 Y_m}{d\beta^2} + d_{w5m} \frac{dY_m}{d\beta} \right) \sin \omega t \qquad (3.14l)$$

Kirchoff shear is given by

$$N_1^* = N_1 + \frac{\partial H}{\partial \beta}$$

$$= \sum_{m=1}^{M} \left(d_{N^*11m} \frac{d^7 Y_m}{d\beta^7} + d_{N^*12m} \frac{d^6 Y_m}{d\beta^6} \right.$$

$$+ d_{N^*13m} \frac{d^5 Y_m}{d\beta^5} + d_{N^*14m} \frac{d^4 Y_m}{d\beta^4} + d_{N^*15m} \frac{d^3 Y_m}{d\beta^3}$$

$$\left. + d_{N^*16m} \frac{d^2 Y_m}{d\beta^2} + d_{N^*17m} \frac{dY_m}{d\beta} + d_{N^*18m} Y_m \right) \sin \omega t$$

$$(3.14m)$$

$$N_2^* = N_2 + \frac{\partial H}{\partial \alpha}$$

$$= \sum_{m=1}^{M} \left(d_{N^*21m} \frac{d^7 Y_m}{d\beta^7} + d_{N^*22m} \frac{d^6 Y_m}{d\beta^6} + \right.$$

$$+ d_{N^*23m} \frac{d^5 Y_m}{d\beta^5} + d_{N^*24m} \frac{d^4 Y_m}{d\beta^4} + d_{N^*25m} \frac{d^3 Y_m}{d\beta^3}$$

$$+ d_{N^*26m} \frac{d^2 Y_m}{d\beta^2} + d_{N^*27m} \frac{dY_m}{d\beta}$$

$$\left. + d_{N^*28m} Y_m \right) \sin \omega t \qquad (3.14n)$$

where

$$d_{T11m} = h \, \Omega \, \frac{1}{B^4} \frac{1}{R_1} X_m$$

$$d_{T12m} = h \, \Omega \, \frac{1}{A^2 B^2} \frac{1}{R_2} \frac{d^2 X_m}{d\alpha^2}$$

$$d_{T21m} = h \, \Omega \, \frac{1}{A^2 B^2} \frac{1}{R_1} \frac{d^2 X_m}{d\alpha^2}$$

$$d_{T22m} = h \, \Omega \, \frac{1}{A^4} \frac{1}{R_2} \frac{d^4 X_m}{d\alpha^4}$$

$$d_{S1m} = - h \, \Omega \, \frac{1}{AB^3} \frac{1}{R_1} \frac{dX_m}{d\alpha} \qquad (3.15)$$

Contd.

$$d_{S2m} = -h \quad \frac{1}{A^3 B} \frac{1}{R_2} \frac{d^3 X_m}{d\alpha^3}$$

$$d_{M11m} = \frac{h^3}{12} \left(B_{12} \frac{1}{B^2} e_5 \right) X_m$$

$$d_{M12m} = \frac{h^3}{12} \left(B_{16} \frac{2}{AB} e_5 + B_{12} \frac{1}{B^2} e_4 \right) \frac{dX_m}{d\alpha}$$

$$d_{M13m} = \frac{h^3}{12} \left(B_{11} \frac{1}{A^2} e_5 + B_{16} \frac{2}{AB} e_4 + B_{12} \frac{1}{B^2} e_3 \right) \frac{d^2 X_m}{d\alpha^2}$$

$$d_{M14m} = \frac{h^3}{12} \left(B_{11} \frac{1}{A^2} e_4 + B_{16} \frac{2}{AB} e_3 + B_{12} \frac{1}{B^2} e_2 \right) \frac{d^3 X_m}{d\alpha^3}$$

$$d_{M15m} = \frac{h^3}{12} \left(B_{11} \frac{1}{A^2} e_3 + B_{16} \frac{2}{AB} e_2 + B_{12} \frac{1}{B^2} e_1 \right) \frac{d^4 X_m}{d\alpha^4}$$

$$d_{M16m} = \frac{h^3}{12} \left(B_{11} \frac{1}{A^2} e_2 + B_{16} \frac{2}{AB} e_1 \right) \frac{d^5 X_m}{d\alpha^5} \qquad (3.15)$$

contd.

$$d_{M17m} = \frac{h^3}{12} (B_{11} \frac{1}{A^2} e_1) \frac{d^6 X_m}{d\alpha^6}$$

$$d_{M21m} = \frac{h^3}{12} (B_{22} \frac{1}{B^2} e_5) X_m$$

$$d_{M22m} = \frac{h^3}{12} (B_{26} \frac{2}{AB} e_5 + B_{22} \frac{1}{B^2} e_4) \frac{dX_m}{d\alpha}$$

$$d_{M23m} = \frac{h^3}{12} (B_{12} \frac{1}{A^2} e_5 + B_{26} \frac{2}{AB} e_4 + B_{22} \frac{1}{B^2} e_3) \frac{d^2 X_m}{d\alpha^2}$$

$$d_{M24m} = \frac{h^3}{12} (B_{12} \frac{1}{A^2} e_4 + B_{26} \frac{2}{AB} e_3 + B_{22} \frac{1}{B^2} e_2) \frac{d^3 X_m}{d\alpha^3}$$

$$d_{M25m} = \frac{h^3}{12} (B_{12} \frac{1}{A^2} e_3 + B_{26} \frac{2}{AB} e_2 + B_{22} \frac{1}{B^2} e_1) \frac{d^4 X_m}{d\alpha^4}$$

$$d_{M26m} = \frac{h^3}{12} (B_{12} \frac{1}{A^2} e_2 + B_{26} \frac{2}{AB} e_1) \frac{d^5 X_m}{d\alpha^5}$$

$$d_{M27m} = \frac{h^3}{12} (B_{12} \frac{1}{A^2} e_1) \frac{d^6 X_m}{d\alpha^6} \qquad (3.15)$$

Contd.

$$d_{H1m} = \frac{h^3}{12} \left(B_{26} \frac{1}{B^2} e_5 \right) X_m$$

$$d_{H2m} = \frac{h^3}{12} \left(B_{66} \frac{2}{AB} e_5 + B_{26} \frac{1}{B^2} e_4 \right) \frac{dX_m}{d\alpha}$$

$$d_{H3m} = \frac{h^3}{12} \left(B_{16} \frac{1}{A^2} e_5 + B_{66} \frac{2}{AB} e_4 \right.$$

$$\left. + B_{26} \frac{1}{B^2} e_3 \right) \frac{d^2 X_m}{d\alpha^2}$$

$$d_{H4m} = \frac{h^3}{12} \left(B_{16} \frac{1}{A^2} e_4 + B_{66} \frac{2}{AB} e_3 \right.$$

$$\left. + B_{26} \frac{1}{B^2} e_2 \right) \frac{d^3 X_m}{d\alpha^3}$$

$$d_{H5m} = \frac{h^3}{12} \left(B_{16} \frac{1}{A^2} e_3 + B_{66} \frac{2}{AB} e_2 \right.$$

$$\left. + B_{26} \frac{1}{B^2} e_1 \right) \frac{d^4 X_m}{d\alpha^4}$$

$$d_{H6m} = \frac{h^3}{12} \left(B_{16} \frac{1}{A^2} e_2 + B_{66} \frac{2}{AB} e_1 \right) \frac{d^5 X_m}{d\alpha^5}$$

$$d_{H7m} = \frac{h^3}{12} \left(B_{16} \frac{1}{A^2} e_1 \right) \frac{d^6 X_m}{d\alpha^6}$$

$$d_{H11m} = -\frac{h^3}{12} \left(B_{26} \frac{1}{B^3} e_5 \right) X_m \qquad (3.15)$$

Contd.

$$d_{N12m} = -\frac{h^3}{12}\left(\left(B_{12} + 2B_{66}\right)\frac{1}{AB^2}e_5 + B_{26}\frac{1}{B^3}e_4\right)\frac{dX_m}{d\alpha}$$

$$d_{N13m} = -\frac{h^3}{12}\left(3B_{16}\frac{1}{A^2B}e_5 + \left(B_{12} + 2B_{66}\right)\frac{1}{AB^2}e_4 + B_{26}\frac{1}{B^3}e_3\right)\frac{d^2X_m}{d\alpha^2}$$

$$d_{N14m} = -\frac{h^3}{12}\left(B_{11}\frac{1}{A^3}e_5 + 3B_{16}\frac{1}{A^2B}e_4 + \left(B_{12} + 2B_{66}\right)\frac{1}{AB^2}e_3 + B_{26}\frac{1}{B^3}e_2\right)\frac{d^3X_m}{d\alpha^3}$$

$$d_{N15m} = -\frac{h^3}{12}\left(B_{11}\frac{1}{A^3}e_4 + 3B_{16}\frac{1}{A^2B}e_3 + \left(B_{12} + 2B_{66}\right)\frac{1}{AB^2}e_2 + B_{26}\frac{1}{B^3}e_1\right)\frac{d^4X_m}{d\alpha^4}$$

$$d_{N16m} = -\frac{h^3}{12}\left(B_{11}\frac{1}{A^3}e_3 + 3B_{16}\frac{1}{A^2B}e_2 + \left(B_{12} + 2B_{66}\right)\frac{1}{AB^2}e_1\right)\frac{d^5X_m}{d\alpha^5}$$

$$d_{N17m} = -\frac{h^3}{12}\left(B_{11}\frac{1}{A^3}e_2 + 3B_{16}\frac{1}{A^2B}e_1\right)\frac{d^6X_m}{d\alpha^6}$$

$$d_{N18m} = -\frac{h^3}{12}\left(B_{11}\frac{1}{A^3}e_1\right)\frac{d^7X_m}{d\alpha^7} \qquad (3.15)$$

Contd.

$$d_{N21m} = -\frac{h^3}{12}\left(B_{22}\frac{1}{B^3}e_5\right)X_m$$

$$d_{N22m} = -\frac{h^3}{12}\left(3B_{26}\frac{1}{AB^2}e_5 + B_{22}\frac{1}{B^3}e_4\right)\frac{dX_m}{d\alpha}$$

$$d_{N23m} = -\frac{h^3}{12}\Big((B_{12} + 2B_{66})\frac{1}{A^2B}e_5 + 3B_{26}\frac{1}{AB^2}e_4$$

$$+ B_{22}\frac{1}{B^3}e_3\Big)\frac{d^2X_m}{d\alpha^2}$$

$$d_{N24m} = -\frac{h^3}{12}\Big(B_{16}\frac{1}{A^3}e_5 + (B_{12} + 2B_{66})\frac{1}{A^2B}e_4$$

$$+ 3B_{26}\frac{1}{AB^2}e_3 + B_{22}\frac{1}{B^3}e_2\Big)\frac{d^3X_m}{d\alpha^3}$$

$$d_{N25m} = -\frac{h^3}{12}\Big(B_{16}\frac{1}{A^3}e_4 + (B_{12} + 2B_{66})\frac{1}{A^2B}e_3$$

$$+ 3B_{26}\frac{1}{AB^2}e_2 + B_{22}\frac{1}{B^3}e_1\Big)\frac{d^4X_m}{d\alpha^4}$$

$$d_{N26m} = -\frac{h^3}{12}\Big(B_{16}\frac{1}{A^3}e_3 + (B_{12} + 2B_{66})\frac{1}{A^2B}e_2$$

$$+ 3B_{26}\frac{1}{AB^2}e_1\Big)\frac{d^5X_m}{d\alpha^5}$$

(3.15)
Contd.

$$d_{N27m} = -\frac{h^3}{12}\left(B_{16}\frac{1}{A^3}e_2 + (B_{12}+2B_{66})\frac{1}{A^2B}e_1\right)\frac{d^6X_m}{d\alpha^6}$$

$$d_{N28m} = -\frac{h^3}{12}\left(B_{16}\frac{1}{A^3}e_1\right)\frac{d^7X_m}{d\alpha^7}$$

$$d_{u1m} = e_4 X_m$$

$$d_{u2m} = e_3\frac{dX_m}{d\alpha}$$

$$d_{u3m} = e_2\frac{d^2X_m}{d\alpha}$$

$$d_{u4m} = e_1\frac{d^3X_m}{d\alpha}$$

$$d_{v1m} = f_4 X_m$$

$$d_{v2m} = f_3\frac{dX_m}{d\alpha}$$

$$d_{v3m} = f_2\frac{d^2X_m}{d\alpha^2}$$

$$d_{v4m} = f_1\frac{d^3X_m}{d\alpha^3}$$

$$d_{w1m} = e_5 X_m$$

(3.15)
Contd.

$$d_{w2m} = c_4 \frac{dX_m}{d\alpha}$$

$$d_{w3m} = c_3 \frac{d^2X_m}{d\alpha^2}$$

$$d_{w4m} = c_2 \frac{d^3X_m}{d\alpha^3}$$

$$d_{w5m} = c_1 \frac{d^4X_m}{d\alpha^4}$$

$$d_{N^*11m} = d_{N11m} + d_{H1m}$$

$$d_{N^*12m} = d_{N12m} + d_{H2m}$$

$$d_{N^*13m} = d_{N13m} + d_{H3m}$$

$$d_{N^*14m} = d_{N14m} + d_{H4m}$$

$$d_{N^*15m} = d_{N15m} + d_{H5m}$$

$$d_{N^*16m} = d_{N16m} + d_{H6m}$$

$$d_{N^*17m} = d_{N17m} + d_{H7m}$$

$$d_{N^*18m} = d_{N18m} \qquad (3.15)$$

Contd.

$$d_{N^*21m} = d_{N21m}$$

$$d_{N^*22m} = d_{N22m} + \frac{d}{d\alpha}(d_{H1m})$$

$$d_{N^*23m} = d_{N23m} + \frac{d}{d\alpha}(d_{H2m})$$

$$d_{N^*24m} = d_{N24m} + \frac{d}{d\alpha}(d_{H3m})$$

$$d_{N^*25m} = d_{N25m} + \frac{d}{d\alpha}(d_{H4m})$$

$$d_{N^*26m} = d_{N26m} + \frac{d}{d\alpha}(d_{H5m})$$

$$d_{N^*27m} = d_{N27m} + \frac{d}{d\alpha}(d_{H6m})$$

$$d_{N^*28m} = d_{N28m} + \frac{d}{d\alpha}(d_{H7m}) \qquad (3.15)$$

3.3 GENERALISED DISPLACEMENTS, FORCES AND MOMENTS

The necessary number of boundary conditions along edges $\beta = 0$ and $\beta = \beta_o$ to solve (3.13) have to be prescribed in terms of the generalised displacements, forces and moments.

Giving virtual displacements $\bar{u}_k(\alpha)$, $\bar{v}_k(\alpha)$ and $\bar{w}_k(\alpha)$ (depending on the boundary conditions) along α, β and γ directions, the corresponding generalised displacements can be written as:

$$U_k = \int u \bar{u}_k \, d\alpha$$

$$= \sum_{m=1}^{M} \left(d_{U1m} \frac{d^3 Y_m}{d\beta^3} + d_{U2m} \frac{d^2 Y_m}{d\beta^2} + d_{U3m} \frac{dY_m}{d\beta} \right.$$

$$\left. + \, d_{U4m} Y_m \right) \sin \omega t$$

$$= \sum_{m=1}^{M} U_{km} \sin \omega t \qquad (3.16a)$$

$$V_k = \int v \, \bar{v}_k \, d\alpha$$

$$= \sum_{m=1}^{M} \left(d_{V1m} \frac{d^3 Y_m}{d\beta^3} + d_{V2m} \frac{d^2 Y_m}{d\beta^2} + d_{V3m} \frac{dY_m}{d\beta} \right.$$

$$\left. + \, d_{V4m} Y_m \right) \sin \omega t$$

$$= \sum_{m=1}^{M} V_{km} \sin \omega t \qquad (3.16b)$$

$$W_k = \int w \, \bar{w}_k \, d\alpha$$

$$= \sum_{m=1}^{M} \left(d_{W1m} \frac{d^4 Y_m}{d\beta^4} + d_{W2m} \frac{d^3 Y_m}{d\beta^3} + d_{W3m} \frac{d^2 Y_m}{d\beta^2} \right.$$

$$\left. + \, d_{W4m} \frac{dY_m}{d\beta} + d_{W5m} Y_m \right) \sin \omega t$$

$$= \sum_{m=1}^{M} W_{km} \sin \omega t \qquad (3.16c)$$

The generalised change in slope in β-direction can be written as

$$w_k' = \int \frac{\partial w}{\partial \beta} \bar{w}_k \, d\alpha$$

$$= \sum_{m=1}^{M} \left(d_{W1m} \frac{d^5 Y_m}{d\beta^5} + d_{W2m} \frac{d^4 Y_m}{d\beta^4} + d_{W3m} \frac{d^3 Y_m}{d\beta^3} \right.$$

$$\left. + \, d_{W4m} \frac{d^2 Y_m}{d\beta^2} + d_{W5m} \frac{dY_m}{d\beta} \right) \sin \omega t \qquad (3.16d)$$

$$(k = 1, 2, \ldots M)$$

where

$$d_{U1m} = \int d_{u1m} \bar{u}_k \, d\alpha$$

$$d_{U2m} = \int d_{u2m} \bar{u}_k \, d\alpha$$

$$d_{U3m} = \int d_{u3m} \bar{u}_k \, d\alpha$$

$$d_{U4m} = \int d_{u4m} \bar{u}_k \, d\alpha$$

$$d_{V1m} = \int d_{v1m} \bar{v}_k \, d\alpha$$

$$d_{V2m} = \int d_{v2m} \bar{v}_k \, d\alpha$$

$$d_{V3m} = \int d_{v3m} \bar{v}_k \, d\alpha$$

$$d_{V4m} = \int d_{v4m} \bar{\bar{v}}_k \, d\alpha$$

$$d_{W1m} = \int d_{w1m} \bar{\bar{w}}_k \, d\alpha$$

$$d_{W2m} = \int d_{w2m} \bar{\bar{w}}_k \, d\alpha$$

$$d_{W3m} = \int d_{w3m} \bar{\bar{w}}_k \, d\alpha$$

$$d_{W4m} = \int d_{w4m} \bar{\bar{w}}_k \, d\alpha$$

$$d_{W5m} = \int d_{w5m} \bar{\bar{w}}_k \, d\alpha \qquad (3.17)$$

All the integrals in (3.16) and (3.17) are taken over the entire range of α.

Now the generalised forces T_k and S_k can be defined as the work done by the forces T_2 and S over the corresponding virtual displacements $\bar{\bar{v}}_k$, $\bar{u}_k$ respectively.

$$S_k = \int S \, \bar{u}_k \, d\alpha$$

$$= \sum_{m=1}^{M} \left(d_{Sk1m} \frac{d^3 Y_m}{d\phi^3} + d_{Sk2m} \frac{dY_m}{d\phi} \right) \sin \omega t$$

$$= \sum_{m=1}^{M} S_{km} \sin \omega t \qquad (3.18a)$$

$$T_k = \int T_2 \bar{\bar{v}}_k \, d\alpha$$

$$= \sum_{m=1}^{M} \left(d_{Tk1m} \frac{d^2 Y_m}{d\beta^2} + d_{Tk2m} Y_m \right) \sin \omega t$$

$$= \sum_{\mu=1}^{M} T_{kM} \sin \omega t \qquad (3.18b)$$

$$(k = 1, 2, \ldots M)$$

where

$$d_{Sk1m} = \int d_{S1m} \bar{\bar{u}}_k \, d\alpha$$

$$d_{Sk2m} = \int d_{S2m} \bar{\bar{u}}_k \, d\alpha$$

$$d_{Tk1m} = \int d_{T21m} \bar{\bar{v}}_k \, d\alpha$$

$$d_{Tk2m} = \int d_{T22m} \bar{\bar{v}}_k \, d\alpha \qquad (3.19)$$

The integrals (3.19) are taken over the entire range of α.

The generalised bending moments M_k, can be expressed as the work done by M_2 (3.14e) over the virtual slope $\bar{\bar{w}}'$ which can be assumed to be equal to $\bar{\bar{w}}_k(\alpha)$ and can be obtained as

$$M_k = \int M_2 \bar{\bar{w}}' d\alpha = M_2 \bar{\bar{w}}_k d\alpha$$

$$= \sum_{m=1}^{M} \left(d_{Mk1m} \frac{d^6 Y_m}{d\beta^6} + d_{Mk2m} \frac{d^5 Y_m}{d\beta^5} + d_{Mk3m} \frac{d^4 Y_m}{d\beta^4} \right.$$

$$+ d_{Mk4m} \frac{d^3 Y_m}{d\beta^3} + d_{Mk5m} \frac{d^2 Y_m}{d\beta^2} + d_{Mk6m} \frac{d Y_m}{d\beta}$$

$$\left. + d_{Mk7m} Y_m \right) \sin \omega t$$

$$= \sum_{m=1}^{M} M_{km} \sin \omega t \tag{3.20}$$

$$(k = 1, 2, \ldots M)$$

where

$$d_{Mk1m} = \int d_{M21m} \bar{\bar{w}}_k d\alpha$$

$$d_{Mk2m} = \int d_{M22m} \bar{\bar{w}}_k d\alpha$$

$$d_{Mk3m} = \int d_{M23m} \bar{\bar{w}}_k d\alpha$$

$$d_{Mk4m} = \int d_{M24m} \bar{\bar{w}}_k d\alpha$$

$$d_{Mk5m} = \int d_{M25m} \bar{\bar{w}}_k d\alpha$$

$$d_{Mk6m} = \int d_{M26m} \bar{\bar{w}}_k d\alpha \tag{3.21}$$

The integrals (3.21) are taken over the entire range of α.

The generalised shear is work done by Kirchoff shear N_2^* (3.14a) on virtual displacement $\bar{\bar{w}}_k$ and is given by

$$N_k^* = \int N_2^* \bar{\bar{w}}_k \, d\alpha$$

$$= \sum_{m=1}^{M} \left(d_{N^*k1m} \frac{d^7 Y_m}{d\beta^7} + d_{N^*k2m} \frac{d^6 Y_m}{d\beta^6} + d_{N^*k3m} \frac{d^5 Y_m}{d\beta^5} \right.$$

$$+ d_{N^*k4m} \frac{d^4 Y_m}{d\beta^4} + d_{N^*k5m} \frac{d^3 Y_m}{d\beta^3} + d_{N^*k6m} \frac{d^2 Y_m}{d\beta^2}$$

$$\left. + d_{N^*k7m} \frac{dY_m}{d\beta} + d_{N^*k8m} Y_m \right) \sin \omega t \qquad (3.22)$$

$$= \sum_{m=1}^{M} N_{km}^* \sin \omega t$$

$$(k = 1, 2, \ldots M)$$

where

$$d_{N^*k1m} = \int d_{N^*21m} \bar{\bar{w}}_k \, d\alpha$$

$$d_{N^*k2m} = \int d_{N^*22m} \bar{\bar{w}}_k \, d\alpha$$

$$d_{N^*k3m} = \int d_{N^*23m} \bar{\bar{w}}_k \, d\alpha$$

$$d_N{}^*{}_{k4m} = \int d_N{}^*{}_{24m}\, \bar{\bar{w}}_k \, d\alpha$$

$$d_N{}^*{}_{k5m} = \int d_N{}^*{}_{25m}\, \bar{\bar{w}}_k \, d\alpha$$

$$d_N{}^*{}_{k6m} = \int d_N{}^*{}_{26m}\, \bar{\bar{w}}_k \, d\alpha$$

$$d_N{}^*{}_{k7m} = \int d_N{}^*{}_{27m}\, \bar{\bar{w}}_k \, d\alpha$$

$$d_N{}^*{}_{k8m} = \int d_N{}^*{}_{28m}\, \bar{\bar{w}}_k \, d\alpha \qquad (3.23)$$

The integrals (3.23) are taken over the entire range of α.

CHAPTER IV

ILLUSTRATION WITH TWO PARALLEL SIDES SIMPLY SUPPORTED

The general solution by Kantorovich method derived in the last chapter is degenerated to Levy type solution for the case when two parallel sides $\alpha = 0$ and $\alpha = \alpha_o$ are simply supported. The boundary conditions along remaining edges are still arbitrary. Let

$$X_m(\alpha) = \sin \frac{m\pi\alpha}{\alpha_o} \tag{4.1}$$

Substitution of (4.1) in (3.13) using orthogonality relations will give the required Levy type solution, as following:

$$A_{mm}\lambda^8 + B_{mm}\lambda^7 + C_{mm}\lambda^6 + D_{mm}\lambda^5 + E_{mm}\lambda^4 + F_{mm}\lambda^3$$

$$+ G_{mm}\lambda^2 + H_{mm}\lambda + I_{mm} = 0 \tag{4.2a}$$

with

$$A_{mm} = \frac{h^2}{12} a_9 \frac{\alpha_o}{2}$$

$$B_{mm} = 0$$

$$C_{mm} = \frac{h^2}{12} a_7 \frac{m^2\pi^2}{\alpha_o^2} \frac{\alpha_o}{2}$$

$$D_{mn} = 0$$

$$E_{mn} = \left(\frac{h^2}{12} a_5 \frac{m^4\pi^4}{\alpha_o^4} + \Omega\, b_3 - \frac{\gamma^*}{g} \omega^2 c_5\right) \frac{\alpha_o}{2}$$

$$F_{mn} = 0$$

$$G_{mn} = \left(- \frac{h^2}{12} a_3 \frac{m^6\pi^6}{\alpha_o^6} - \Omega\, b_2 \frac{m^2\pi^2}{\alpha_o^2} + \frac{\gamma^*}{g} \omega^2 c_3 \frac{m^2\pi^2}{\alpha_o^2}\right)\frac{\alpha_o}{2}$$

$$H_{mn} = 0$$

$$I_{mn} = \left(\frac{h^2}{12} a_1 \frac{m^8\pi^8}{\alpha_o^8} + \Omega\, b_1 \frac{m^4\pi^4}{\alpha_o^4} - \frac{\gamma^*}{g} \omega^2 c_1 \frac{m^4\pi^4}{\alpha_o^4}\right) \frac{\alpha_o}{2}$$

(4.2b)

Hence (4.2) can be written as

$$A_{1mn}\lambda^8 + A_{2mn}\lambda^6 + A_{3mn}\lambda^4 + A_{4mn}\lambda^2 + A_{5mn} = 0 \tag{4.3a}$$

$$(m = 1, 2, \ldots M)$$

where

$$A_{1mn} = \frac{h^2}{12} a_9$$

$$A_{2mn} = - \frac{h^2}{12} a_7 \frac{m^2\pi^2}{\alpha_o^2}$$

$$A_{3mn} = \frac{h^2}{12} a_3 \frac{m^4\pi^4}{\alpha_o^4} + \Omega\, b_3 - \frac{\gamma^*}{g} \omega^2 e_5$$

$$A_{4mn} = -\frac{h^2}{12} a_3 \frac{m^6\pi^6}{\alpha_o^6} - \Omega\, b_2 \frac{m^2\pi^2}{\alpha_o^2} + \frac{\gamma^*}{g} \omega^2 e_3 \frac{m^2\pi^2}{\alpha_o^2}$$

$$A_{5mn} = \frac{h^2}{12} a_1 \frac{m^8\pi^8}{\alpha_o^8} + \Omega\, b_1 \frac{m^4\pi^4}{\alpha_o^4} - \frac{\gamma^*}{g} \omega^2 e_1 \frac{m^4\pi^4}{\alpha_o^4}$$

(4.3b)

4.1 CHARACTERISTIC ROOTS

The solutions of (4.3a) depend upon the coefficients (4.3b) and are given as follows|35|:

Case 1: $\pm n_1$, $\pm n_2$, $\pm n_3$, $\pm n_4$

Case 2: $\pm n_1$, $\pm n_2$, $\pm n_3$, $\pm in_4$

Case 3: $\pm n_1$, $\pm n_2$, $\pm in_3$, $\pm in_4$

Case 4: $\pm n_1$, $\pm in_2$, $\pm in_3$, $\pm in_4$

Case 5: $\pm in_1$, $\pm in_2$, $\pm in_3$, $\pm in_4$

Case 6: $\pm n_1$, $\pm n_2$, $\pm (n_3 \pm in_4)$

Case 7: $\pm n_1$, $\pm in_2$, $\pm (n_3 \pm in_4)$

Case 8: $\pm in_1$, $\pm in_2$, $\pm (n_3 \pm in_4)$

Case 9: $\pm (n_1 \pm in_2)$, $\pm (n_3 \pm in_4)$ (4.4)

where n_1, n_2, n_3 and n_4 are components of the characteristic roots.

With the help of (4.4), the function $Y_m(\beta)$ from (3.12) can be written as

$$Y_m = C_{1m} Y_{1m} + C_{2m} Y_{2m} + C_{3m} Y_{3m} + C_{4m} Y_{4m} + C_{5m} Y_{5m} + C_{6m} Y_{6m} + C_{7m} Y_{7m} + C_{8m} Y_{8m} \quad (4.5)$$

where C_{1m} through C_{8m} are arbitrary constants to be evaluated from the boundary conditions along $\beta = 0$ and $\beta = \beta_o$. (4.5) can be written in the matrix form as

$$Y_m = \{C\}^T \{Y_R\} \quad (4.6)$$

The superscript 'T' denotes the transpose of the matrix and

$$C = \begin{Bmatrix} C_{1m} \\ C_{2m} \\ C_{3m} \\ C_{4m} \\ C_{5m} \\ C_{6m} \\ C_{7m} \\ C_{8m} \end{Bmatrix} \quad (4.7a)$$

$$Y_R = \begin{Bmatrix} Y_{1m} \\ Y_{2m} \\ Y_{3m} \\ Y_{4m} \\ Y_{5m} \\ Y_{6m} \\ Y_{7m} \\ Y_{8m} \end{Bmatrix} \qquad (4.7b)$$

The differential relations of Y_R can be expressed as

$$\frac{d^n\{Y_R\}}{d\beta^n} = [N_R]^n \{Y_R\} \qquad (4.8)$$

where $[N_R]$ is a matrix depending upon the characteristic roots (4.4). Also

$$[N_R]^n = [N_R]\,[N_R] \;\; .... \; n \text{ times}$$

From (4.6) and(4.8), it can be seen that

$$\frac{d^n Y_m}{d\beta^n} = \{C\}^T [N_R]^n \{Y_R\} \qquad (4.9)$$

With the help of (3.12) and (4.4) the basis of solutions $\{Y_R\}$ is given below as a row matrix $\{Y_R\}^T$, along with the matrix $[N_R]$ for various cases.

Case: 1:

$$\{Y_R\}^T = \{\cosh n_1\beta,\ \sinh n_1\beta,\ \cosh n_2\beta,\ \sinh n_2\beta$$

$$\cosh n_3\beta,\ \sinh n_3\beta,\ \cosh n_4\beta,\ \sinh n_4\beta\ \}$$

(4.10a)

$$[N_R] = \begin{bmatrix} 0 & d_1n_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ n_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d_2n_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & n_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & d_3n_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & n_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_4n_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & n_4 & 0 \end{bmatrix}$$

(4.10b)

where

$$d_1 = d_2 = d_3 = d_4 = 1$$

Case 2:

$$\{Y_R\}^T = \Big\{\cosh n_1\beta,\ \sinh n_1\beta,\ \cosh n_2\beta,\ \sinh n_2\beta,$$

$$\cosh n_3\beta,\ \sinh n_3\beta,\ \cos n_4\beta,\ \sin n_4\beta\Big\} \qquad (4.11a)$$

$[N_R]$ is given by (4.10b) with

$$d_1 = d_2 = d_3 = 1 \text{ and } d_4 = -1 \qquad (4.11b)$$

Case 3:

$$\{Y_R\}^T = \Big\{\cosh n_1\beta,\ \sinh n_1\beta,\ \cosh n_2\beta,\ \sinh n_2\beta,$$

$$\cos n_3\beta,\ \sin n_3\beta,\ \cos n_4\beta,\ \sin n_4\beta\Big\} \qquad (4.12a)$$

$[N_R]$ is given by (4.10b) with

$$d_1 = d_2 = 1 \text{ and } d_3 = d_4 = -1 \qquad (4.12b)$$

Case 4:

$$\{Y_R\}^T = \Big\{\cosh n_1\beta,\ \sinh n_1\beta,\ \cos n_2\beta,\ \sin n_2\beta,$$

$$\cos n_3\beta,\ \sin n_3\beta,\ \cos n_4\beta,\ \sin n_4\beta\Big\} \qquad (4.13a)$$

$[N_R]$ is given by (4.10b) with

$$d_1 = 1 \text{ and } d_2 = d_3 = d_4 = -1 \qquad (4.13b)$$

Case 5:

$$\{Y_R\}^T = \Big\{\cos n_1\beta,\ \sin n_1\beta,\ \cos n_2\beta,\ \sin n_2\beta,$$

$$\cos n_3\beta,\ \sin n_3\beta,\ \cos n_4\beta,\ \sin n_4\beta\Big\} \tag{4.14a}$$

$[N_R]$ is given by (4.10b) with

$$d_1 = d_2 = d_3 = d_4 = -1 \tag{4.14b}$$

Case 6:

$$\{Y_R\}^T = \Big\{\cosh n_1\beta,\ \sinh n_1\beta,\ \cosh n_2\beta,\ \sinh n_2\beta,$$

$$\cosh n_3\beta \cos n_4\beta,\ \sinh n_3\beta \cos n_4\beta,$$

$$\sinh n_3\beta \sin n_4\beta,\ \cosh n_3\beta \sin n_4\beta\Big\} \tag{4.15a}$$

$$[N_R] = \begin{bmatrix} 0 & d_1n_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ n_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d_2n_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & n_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & n_3 & 0 & -n_4 \\ 0 & 0 & 0 & 0 & n_3 & 0 & -n_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & n_4 & 0 & n_3 \\ 0 & 0 & 0 & 0 & n_4 & 0 & n_3 & 0 \end{bmatrix} \tag{4.15b}$$

Case 7:

$$\{Y_R\}^T = \Big\{\cosh n_1\beta,\ \sinh n_1\beta,\ \cos n_2\beta,\ \sin n_2\beta,$$

$$\cosh n_3\beta \cos n_4\beta,\ \sinh n_3\beta \cos n_4\beta,$$

$$\sinh n_3\beta \sin n_4\beta,\ \cosh n_3\beta \sin n_4\beta\Big\} \tag{4.16a}$$

$[N_R]$ is given by (4.15b) with

$$d_1 = 1 \quad \text{and} \quad d_2 = -1 \tag{4.16b}$$

Case 8:

$$\{Y_R\}^T = \Big\{\cos n_1\beta,\ \sin n_1\beta,\ \cos n_2\beta,\ \sin n_2\beta,$$

$$\cosh n_3\beta \cos n_4\beta,\ \sinh n_3\beta \cos n_4\beta,$$

$$\sinh n_3\beta \sin n_4\beta,\ \cosh n_3\beta \sin n_4\beta\Big\} \tag{4.17a}$$

$[N_R]$ is given by (4.15b) with

$$d_1 = d_2 = -1 \tag{4.17b}$$

Case 9:

Case 9:

$$\{Y_R\}^T = \{ \cosh n_1\beta \cos n_2\beta,\ \sinh n_1\beta \cos n_2\beta,$$

$$\sinh n_1\beta \sin n_2\beta,\ \cosh n_1\beta \sin n_2\beta,$$

$$\cosh n_3\beta \cos n_4\beta,\ \sinh n_3\beta \cos n_4\beta,$$

$$\sinh n_3\beta \sin n_4\beta,\ \cosh n_3\beta \sin n_4\beta \}$$

(4.18a)

$$[N_R] = \begin{bmatrix} 0 & n_1 & 0 & -n_2 & 0 & 0 & 0 & 0 \\ n_1 & 0 & -n_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & n_2 & 0 & n_1 & 0 & 0 & 0 & 0 \\ n_2 & 0 & n_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & n_3 & 0 & -n_4 \\ 0 & 0 & 0 & 0 & n_3 & 0 & -n_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & n_4 & 0 & n_3 \\ 0 & 0 & 0 & 0 & n_4 & 0 & n_3 & 0 \end{bmatrix}$$

(4.18b)

4.2 NATURAL FREQUENCIES AND MODE SHAPES

By choosing orthogonal set of functions as

$$\bar{u}_k(\alpha) = \cos \frac{n\pi\alpha}{\alpha_o}$$

$$\bar{v}_k(\alpha) = \sin \frac{m\pi\alpha}{\alpha_o}$$

$$\bar{w}_k(\alpha) = \sin \frac{m\pi\alpha}{\alpha_o} \qquad (4.19)$$

for the case under consideration, all the parameters U_k, V_k, W_k, W_k, S_k, T_k, M_k and M_k^* can be reduced to a system of independent equations for each harmonic (value of m). Hence for m = k,

$$U_k = U_m = U_{mm}$$

$$V_k = V_m = V_{mm}$$

$$W_k = W_m = W_{mm}$$

$$W_k = W_m = W_{mm}$$

$$S_k = S_m = S_{mm}$$

$$T_k = T_m = T_{mm}$$

$$M_k = M_m = M_{mm}$$

$$M_k^* = M_m^* = M_{mm}^* \qquad (4.20)$$

Using (4.19) in (3.16), (3.18), (3.20) and (3.22), it can be seen that

$$U_m = \left[d_{U2m}\ [N_R]^2 \{Y_R\} + d_{U4m} \{Y_R\} \right]^T \{C\}$$

$$V_m = \left[d_{V1m}\ [N_R]^3 \{Y_R\} + d_{V3m}\ [N_R] \{Y_R\} \right]^T \{C\}$$

$$W_m = \left[d_{W1m}\ [N_R]^4 \{Y_R\} + d_{W3m}\ [N_R]^2 \{Y_R\} + d_{W5m} \{Y_R\} \right]^T \{C\}$$

$$W_m' = \left[d_{W1m}\ [N_R]^5 \{Y_R\} + d_{W3m}\ [N_R]^3 \{Y_R\} + d_{W5m}\ [N_R] \{Y_R\} \right]^T \{C\}$$

$$S_m = \left[d_{Sk1m}\ [N_R]^3 \{Y_R\} + d_{Sk2m}\ [N_R] \{Y_R\} \right]^T \{C\}$$

$$T_m = \left[d_{Tk1m}\ [N_R]^2 \{Y_R\} + d_{Tk2m} \{Y_R\} \right]^T \{C\}$$

$$M_m = \left[d_{Mk1m}\ [N_R]^6 \{Y_R\} + d_{Mk3m}\ [N_R]^4 \{Y_R\} + d_{Mk5m}\ [N_R]^2 \{Y_R\} + d_{Mk7m} \{Y_R\} \right]^T \{C\}$$

$$M_m^* = \left[d_{M^*k1m}\ [N_R]^7 \{Y_R\} + d_{M^*k3m}\ [N_R]^5 \{Y_R\} + d_{M^*k5m}\ [N_R]^3 \{Y_R\} + d_{M^*k7m}\ [N_R] \{Y_R\} \right]^T \{C\}$$

(4.21)

Here a factor $(\frac{\varepsilon_o}{2} \sin \omega t)$ common to all is not entered since it will be cancelled throughout due to homogeneous solution.

By choosing proper boundary conditions for the above generalised displacements, forces, moments and shear along $\beta = 0$ and $\beta = \beta_o$, a set of homogeneous equations can be constructed in the matrix form. By making determinant zero, natural frequency is obtained. For mode shapes, the third of (4.21) is used.

4.3 BOUNDARY CONDITIONS

The three fundamental cases are considered

(1) Shell with Simple Support:

Assuming the sides $\beta = 0$ and $\beta = \beta_o$ as simply supported, the boundary conditions can be stated as follows:

$$\begin{aligned} U_m &= 0 \\ W_m &= 0 \\ T_m &= 0 \\ M_m &= 0 \end{aligned} \qquad (4.22)$$

(2) Shell with Fixed or Built in Supports:

For sides $\beta = 0$ and $\beta = \beta_o$ as fixed, the boundary conditions can be stated as:

$$U_m = 0$$

$$V_m = 0$$

$$W_m = 0 \qquad (4.23)$$

$$W_m = 0$$

(3) Shell with Free Sides:

For this case, boundary conditions will be:

$$S_m = 0$$

$$T_m = 0$$

$$M_m = 0 \qquad (4.24)$$

$$N^*_m = 0$$

CHAPTER V

NUMERICAL RESULTS AND CONCLUSIONS

5.1 CLOSED FORM SOLUTION

The derivations and procedure outlined so far is checked by assuming the closed form solution for the type of B.C. (1) (4.22).

Let

$$\phi = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{nm} \sin \lambda_n \alpha \sin \mu_m \beta \sin \omega_{nm} t$$

$$w = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} B_{nm} \sin \lambda_n \alpha \sin \mu_m \beta \sin \omega_{nm} t \tag{5.1}$$

where

$$\lambda_n = \frac{n\pi}{\alpha_0}$$

$$\mu_m = \frac{m\pi}{\beta_0} \tag{5.2}$$

Substituting (5.1) in (2.16)and (2.17) and using orthogonality relations, A_{nm} and B_{nm} can be eliminated. This leads to the frequency of free vibrations of shells with constant curvatures.

$$\omega_{mn}^2 = \frac{g}{\gamma^* h}\left[\frac{h^3}{12}\left(d_1 \lambda_n^4 + d_3 \lambda_n^2 \mu_m^2 + d_5 \mu_m^4\right) + \frac{h\,\Omega\left(\frac{1}{R_1}\frac{1}{B^2}\mu_m^2 + \frac{1}{R_2}\frac{1}{A^2}\lambda_n^2\right)^2}{e_1 \lambda_n^4 + e_3 \lambda_n^2 \mu_m^2 + e_5 \mu_m^4}\right] \quad (5.3)$$

5.2 CYLINDRICAL SHELL

The equation (5.3) gives frequencies of free vibrations of cylindrical shells for $R_1 = \infty$. The ratio a/b in Figure (5.1) is varied from 0.5 to 2.0 for various values of R/b. The variation of non-dimensionalised frequency ratio ω^* from equation (A-1) is presented in Figures (5.2) through (5.8) for various parameters. The material considered is isotropic for which elastic constants are given by

$$B_{11} = \frac{E}{1 - \nu^2}$$

$$B_{22} = \frac{E}{1 - \nu^2}$$

$$B_{12} = \frac{E}{1 - \nu^2}$$

$$B_{66} = \frac{E}{2(1 + \nu)}$$

$$B_{16} = B_{26} = 0$$

ν, the poisson's ratio, is assumed to be 0.17.

It is seen that the lowest frequency is given by two, three or four half sine waves (i.e. ω^*_{12}, ω^*_{13}, ω^*_{14} respectively) along lateral (β) direction as a/b is varied from 2.0 to 0.5 for different H/b. But ω^*_{11} is never lowest for the range of a/b considered. It may become lowest if a/b is increased beyond 2.0. This behaviour is also observed by Onishvili |34|.

These results are confirmed by the procedure developed in this work and are presented in Figure (5.9). Typical eigen vectors are shown in Figure (5.10).

The variation of ω^* for type of B.C. (3) of cylindrical shell is shown in Figure (5.11). It is seen that the frequencies for this case are generally lower than those for type of B.C. (1). The mode shapes for the fundamental frequencies are shown in figure (5.12).

5.3 DOUBLY CURVED SHELLS

Free vibration frequencies of doubly curved shells of isotropic material are shown in Figure (5.13) for type of B.C. (1) and in Figure (5.14) for type of B.C. (3). This variation of frequency with respect to shell parameters is generally similar.

As an illustration of the method following typical orthotropic materials are chosen. Their elastic constants are taken from reference |41| and are presented below.

Elastic Constants for Ply Wood, Unit = 10^6 lb/in^2:

	B_{11}	B_{22}	B_{12}	B_{66}	B_{16}	B_{26}
1. Maple 5 - ply	1.87	0.60	0.073	0.159	0	0
2. Afara 3 - ply	1.96	0.165	0.043	0.110	0	0

It can be seen that the fundamental frequencies for both the materials considered are smaller than those for isotropic one as can be observed from Figures (5.15) to (5.18). Moreover frequencies given by Afara 3-ply are smaller than those give by Maple 5-ply. Hence it can be said that they are sensitive to the changes in values of elastic constants.

COORDINATE SYSTEM

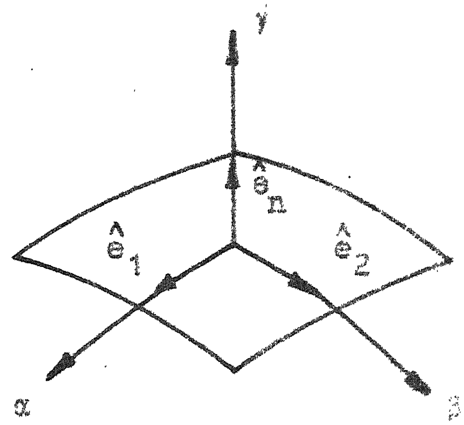


Figure 2.1

DISPLACEMENT AND CURVATURES OF A POINT

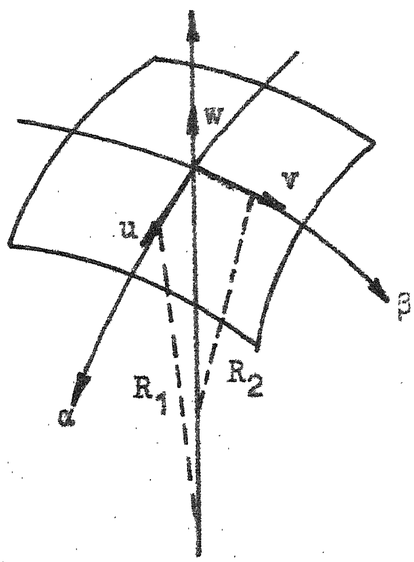
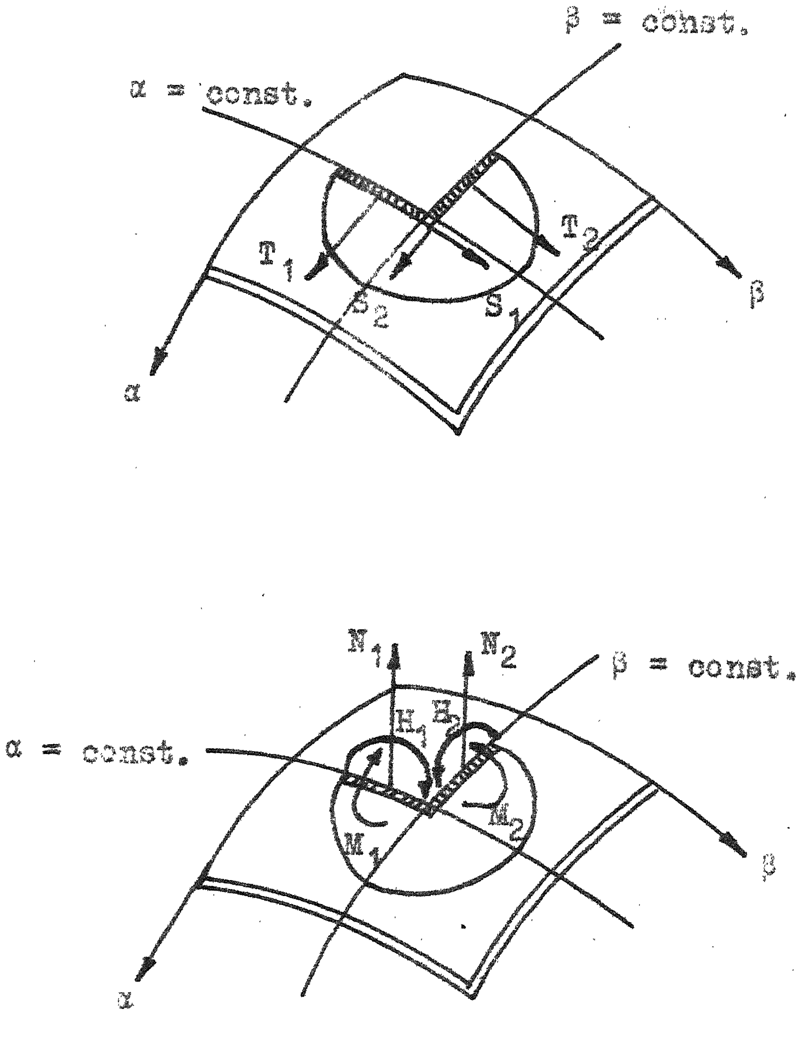


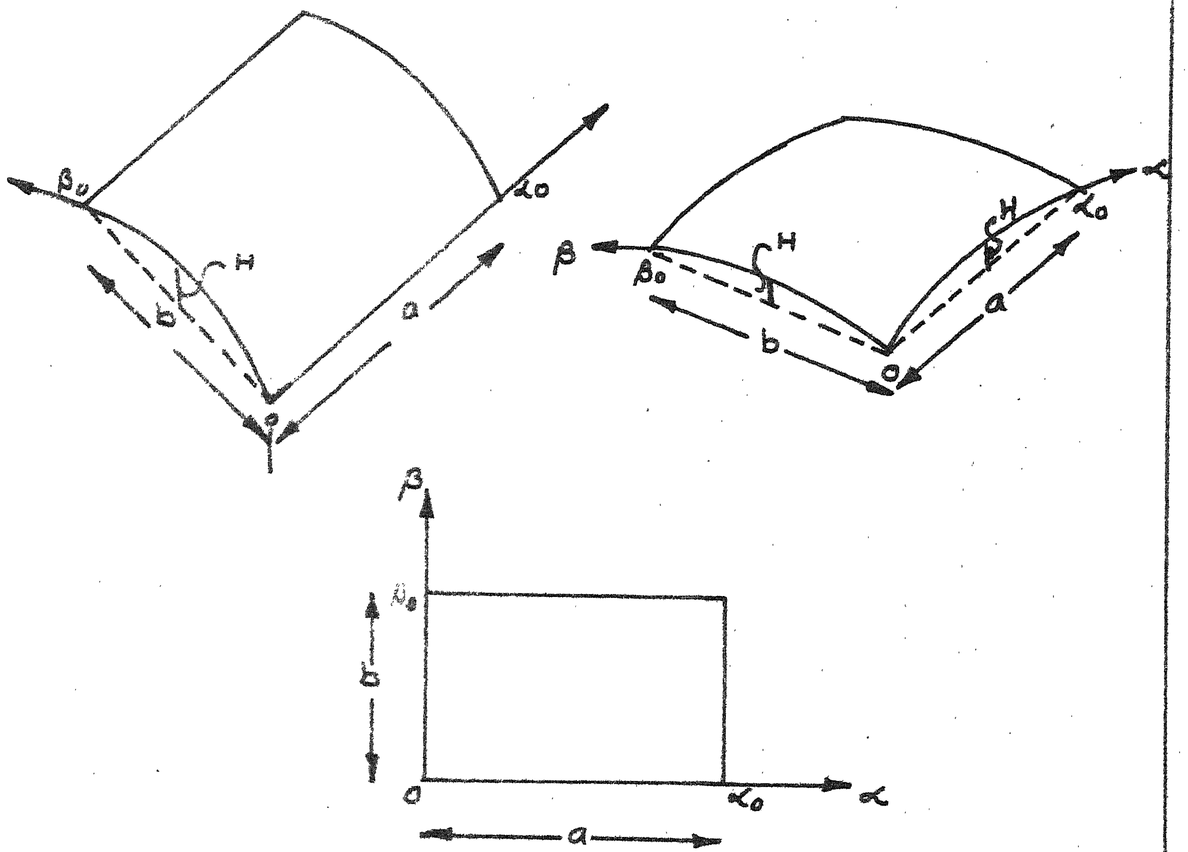
Figure 2.2



FORCES AND MOMENTS ACTING AT A POINT

Figure 2.3

TYPES OF BOUNDARY CONDITIONS



Plan

$\alpha = 0$ and $\alpha = \alpha_o$ Simply Supported Edges

$\beta = 0$ and $\beta = \beta_o$ Arbitrary

TYPES OF BOUNDARY CONDITIONS (B.C.)

(1) $\beta = 0$ and $\beta = \beta_o$ Simply Supported

(2) $\beta = 0$ and $\beta = \beta_o$ Fixed

(3) $\beta = 0$ and $\beta = \beta_o$ Free

Figure 5.1

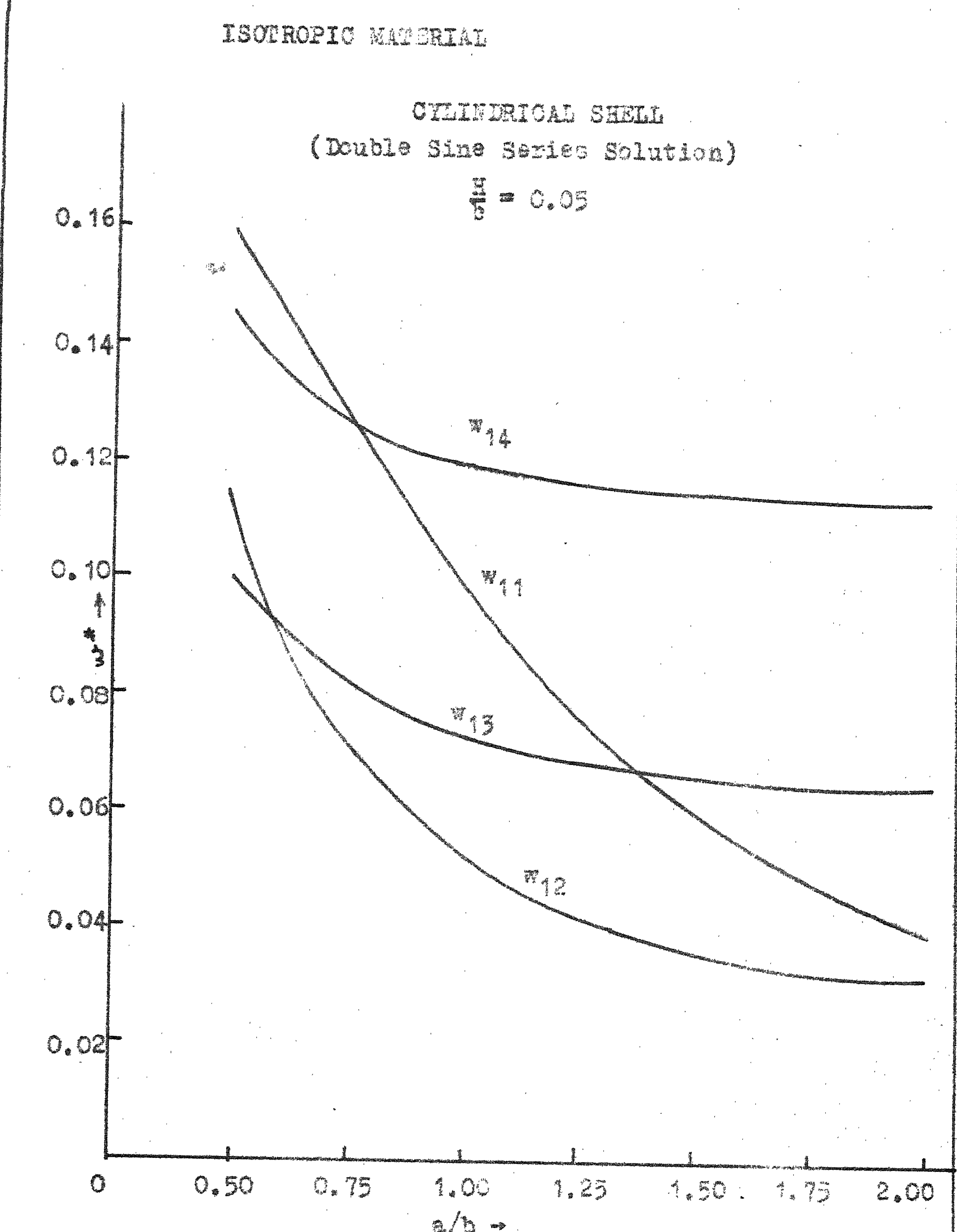

ISOTROPIC MATERIAL
CYLINDRICAL SHELL
(Double Sine Series Solution)
H/b = 0.05
0.16
0.14
0.12
0.10
0.08
0.06
0.04
0.02
w₃*
w14
w11
w13
w12
0
0.50
0.75
1.00
1.25
1.50
1.75
2.00
a/b →

Figure 5.2

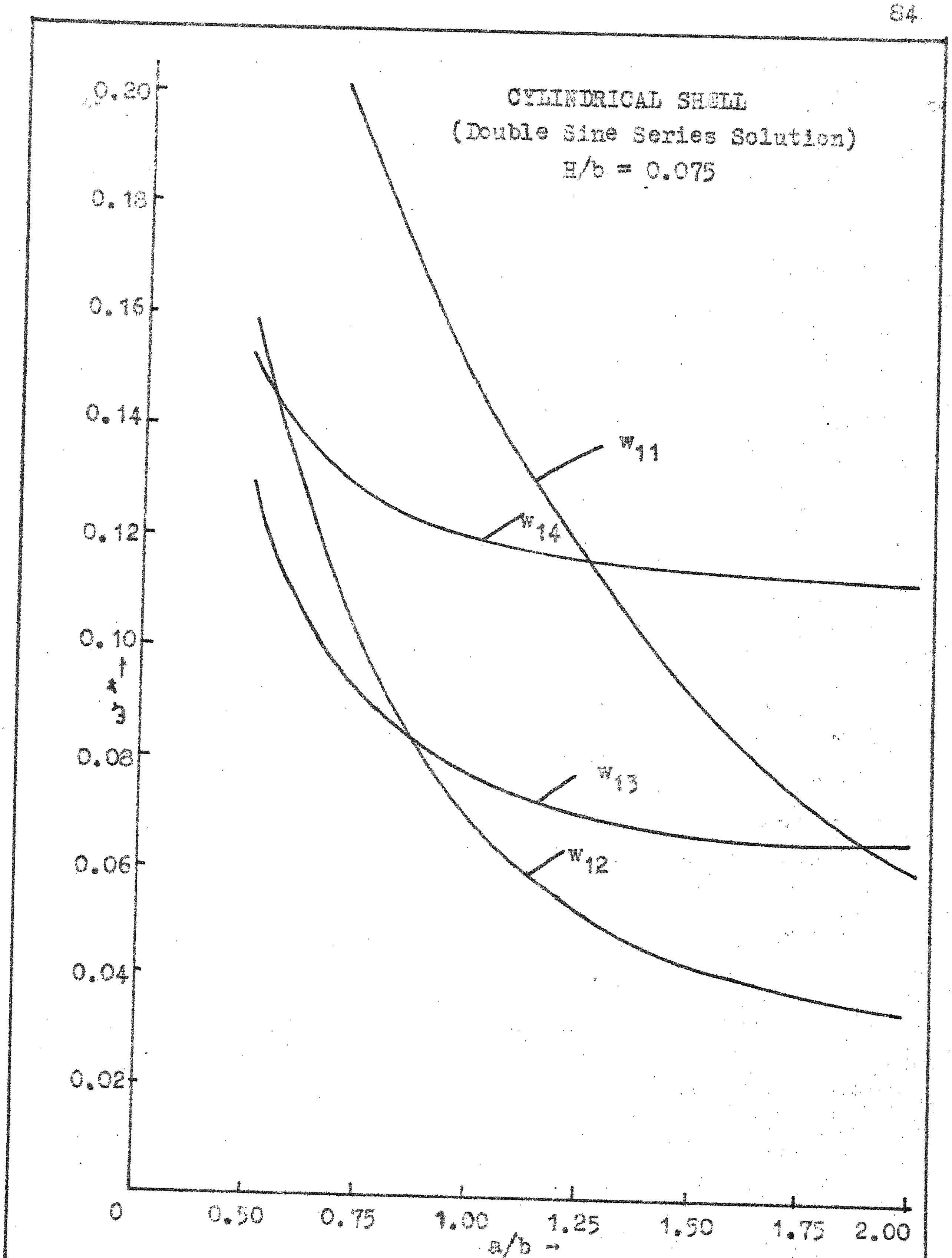
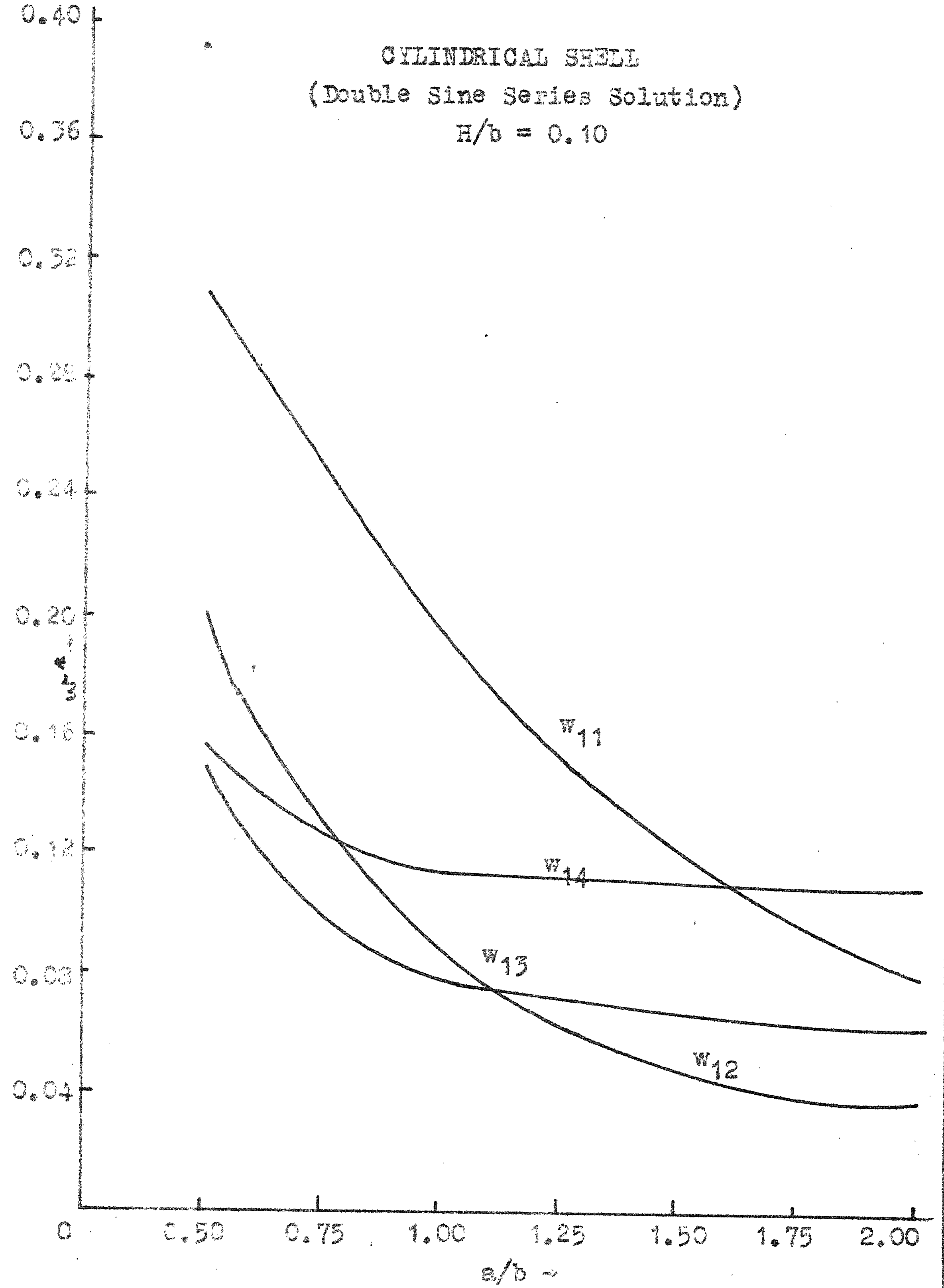

CYLINDRICAL SHELL
(Double Sine Series Solution)
H/b = 0.075
0.20
0.18
0.16
0.14
0.12
0.10
0.08
0.06
0.04
0.02
0
0.50
0.75
1.00
1.25
1.50
1.75
2.00
a/b →
w11
w14
w13
w12

Figure 5.3



CYLINDRICAL SHELL
(Double Sine Series Solution)
H/b = 0.10
0.40
0.36
0.32
0.28
0.24
0.20
0.16
0.12
0.08
0.04
0
0.50
0.75
1.00
1.25
1.50
1.75
2.00
a/b →
w_{11}
w_{14}
w_{13}
w_{12}

Figure 5.4

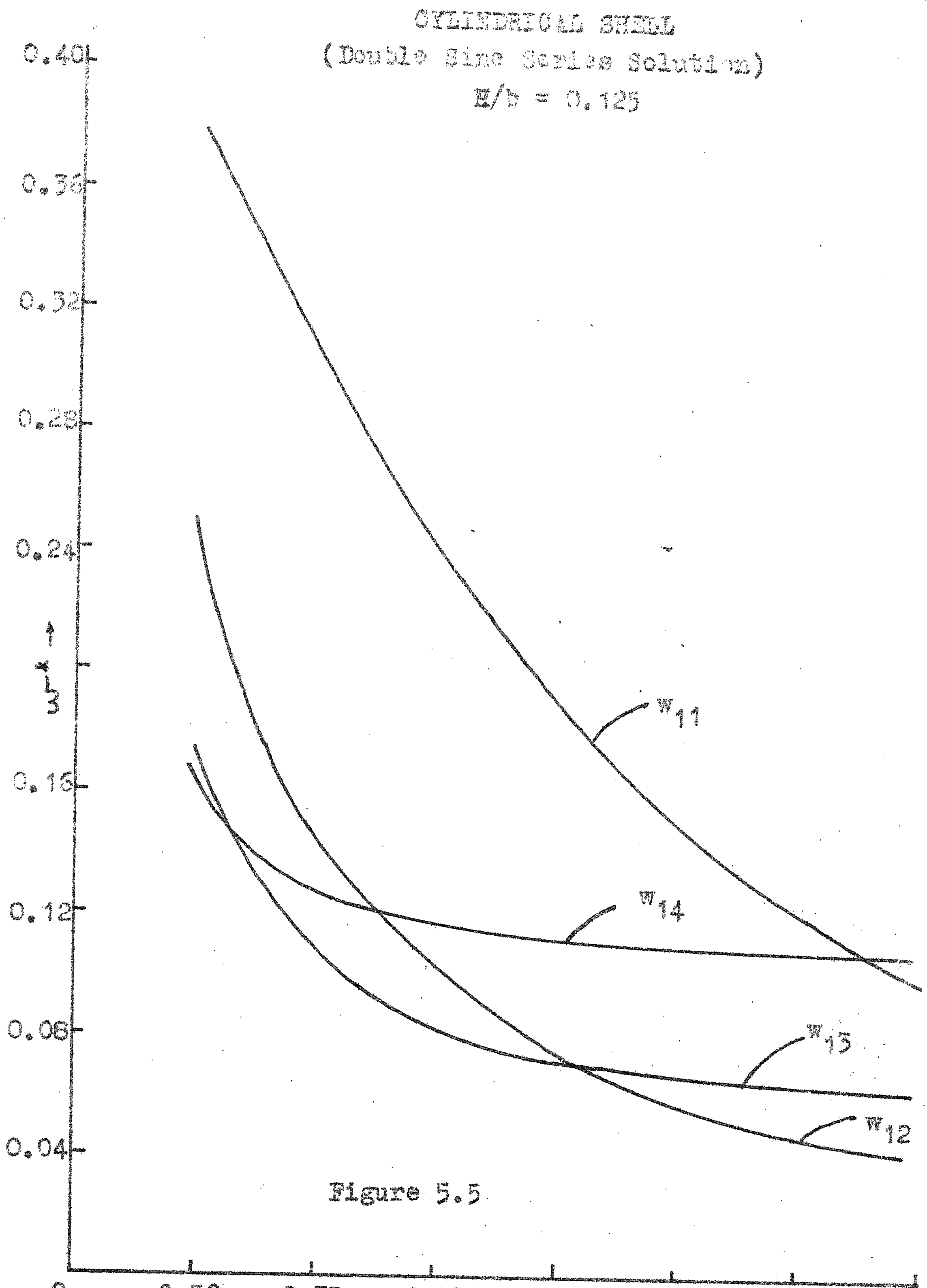
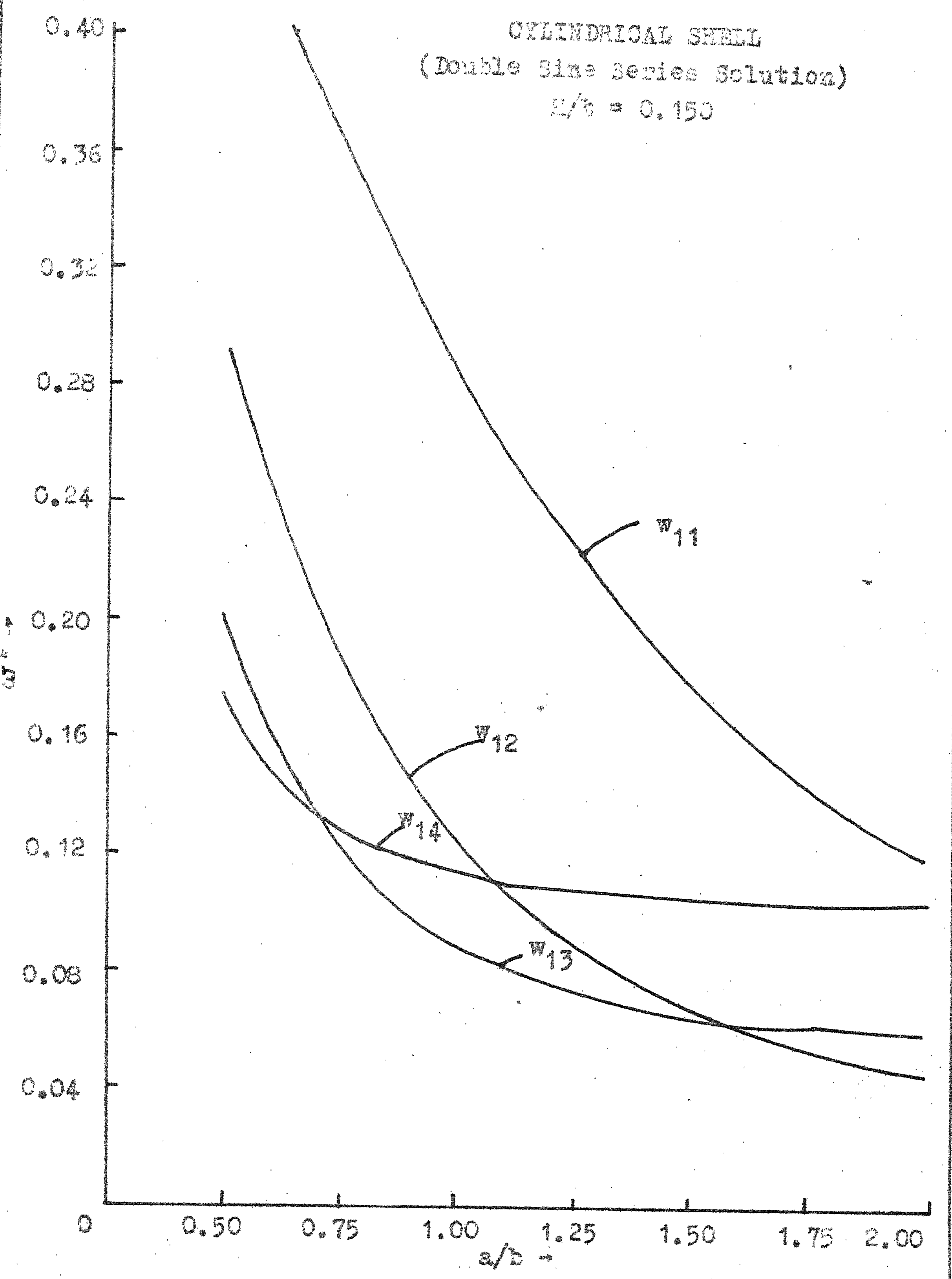

CYLINDRICAL SHELL
(Double Sine Series Solution)
H/b = 0.125
0.40
0.36
0.32
0.28
0.24
0.16
0.12
0.08
0.04
0
0.50
0.75
1.00
a/b
1.25
1.50
1.75
2.00
w_{11}
w_{14}
w_{13}
w_{12}

Figure 5.5



CYLINDRICAL SHELL
(Double Sine Series Solution)
H/b = 0.150
0.40
0.36
0.32
0.28
0.24
0.20
0.16
0.12
0.08
0.04
0
ω^*
w_{11}
w_{12}
w_{13}
w_{14}
0.50
0.75
1.00
1.25
1.50
1.75
2.00
a/b →

Figure 5.6

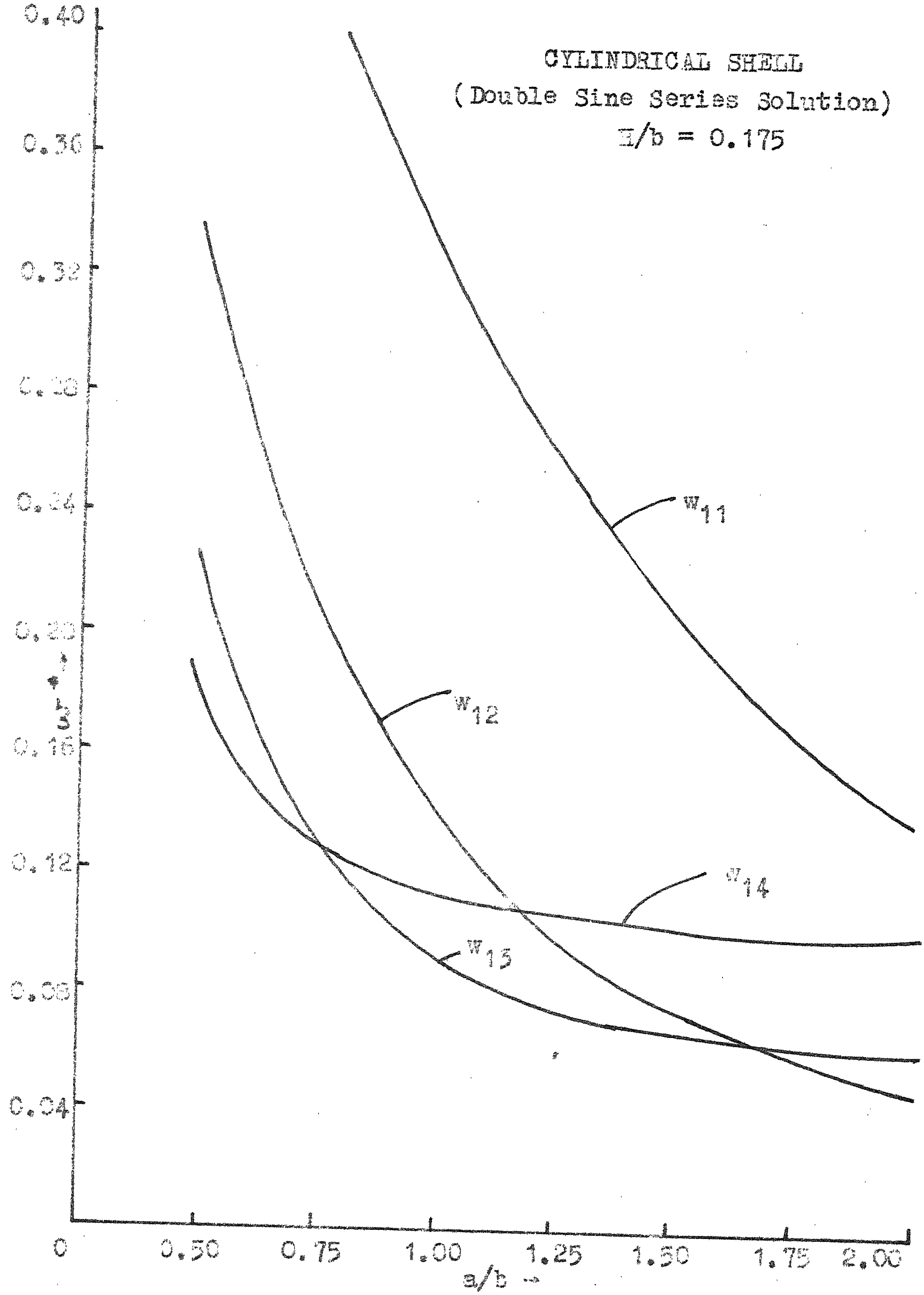

CYLINDRICAL SHELL
(Double Sine Series Solution)
H/b = 0.175
0.40
0.36
0.32
0.28
0.24
0.20
0.16
0.12
0.08
0.04
0
0.50
0.75
1.00
1.25
1.50
1.75
2.00
a/b →
w_{11}
w_{12}
w_{14}
w_{15}

Figure 5.7

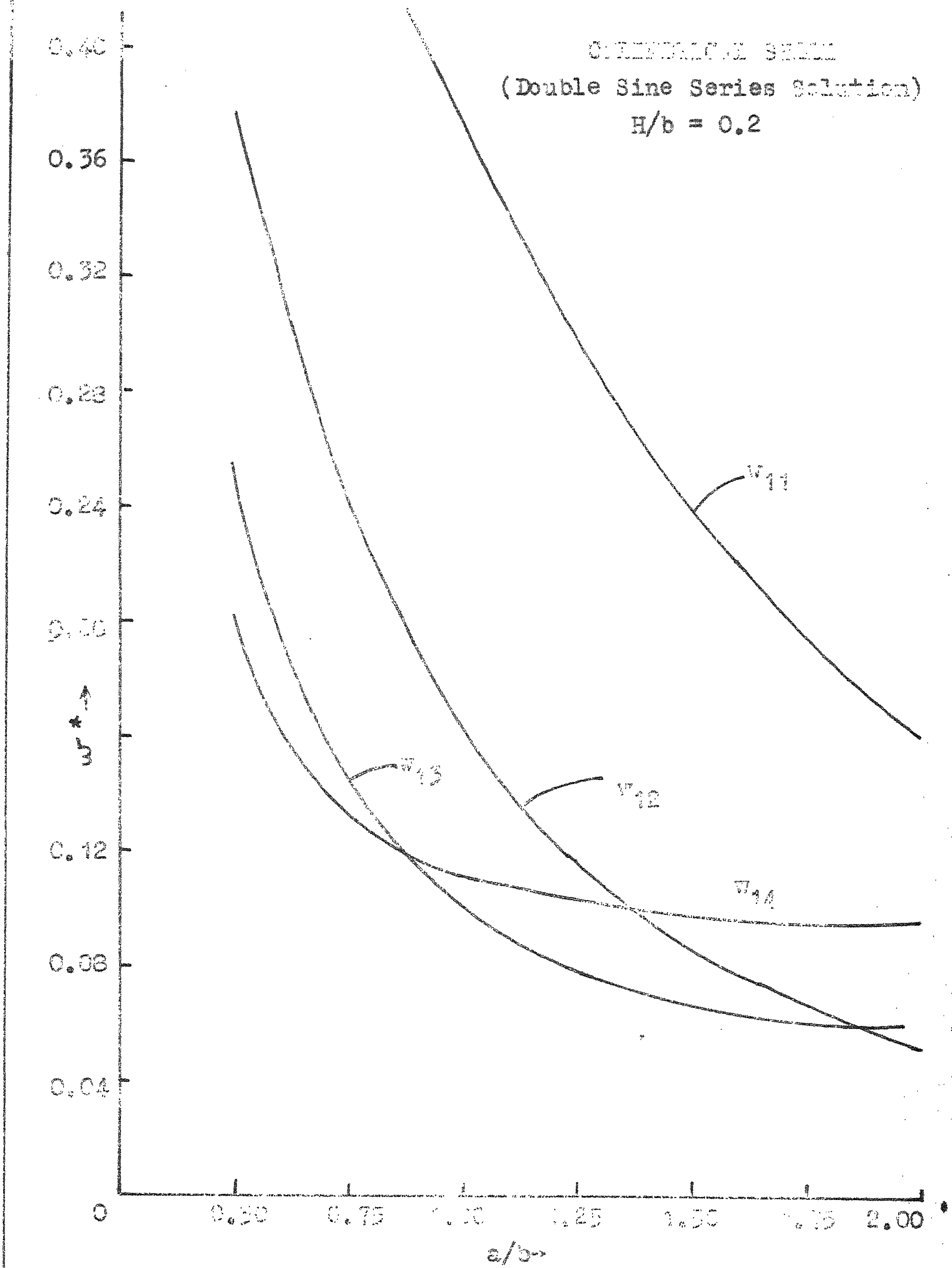

(Double Sine Series
H/b = 0.2
0.40
0.36
0.32
0.28
0.24
0.12
0.08
0.04
0
0.50
0.75
1.25
1.50
2.00
a/b→
w^*
W_{11}
W_{12}
W_{13}
W_{14}

Figure 5.8

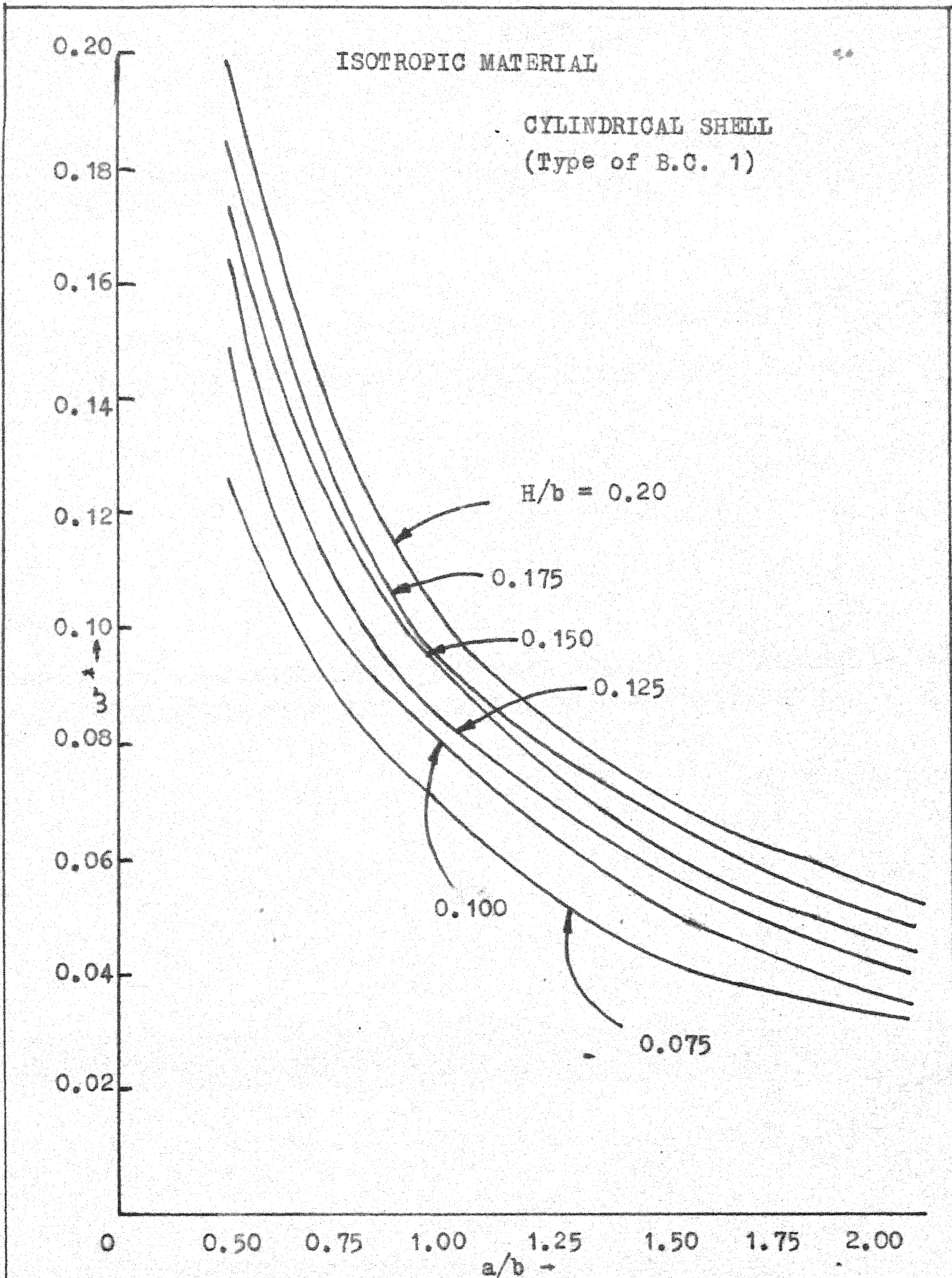
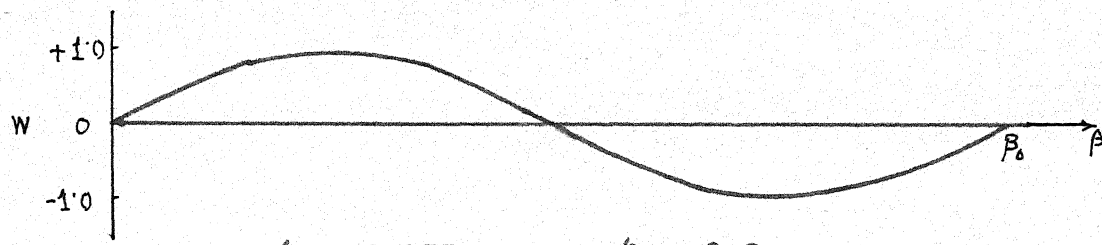
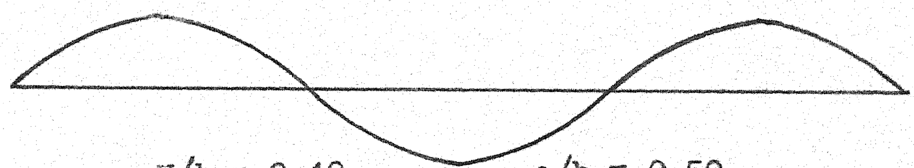

ISOTROPIC MATERIAL
CYLINDRICAL SHELL
(Type of B.C. 1)
H/b = 0.20
0.175
0.150
0.125
0.100
0.075
0.20
0.18
0.16
0.14
0.12
0.10
0.08
0.06
0.04
0.02
0
0.50
0.75
1.00
1.25
1.50
1.75
2.00
a/b →

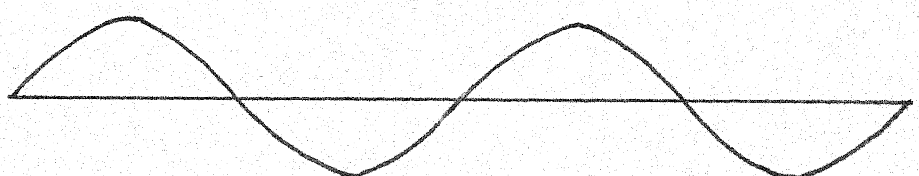
Figure 5.9



H/b = 0.075 a/b = 2.0



H/b = 0.10 a/b = 0.50



H/b = 0.125 a/b = 0.50

TYPICAL EIGEN VECTORS
(For Type of B.C. 1)

Figure 5.10

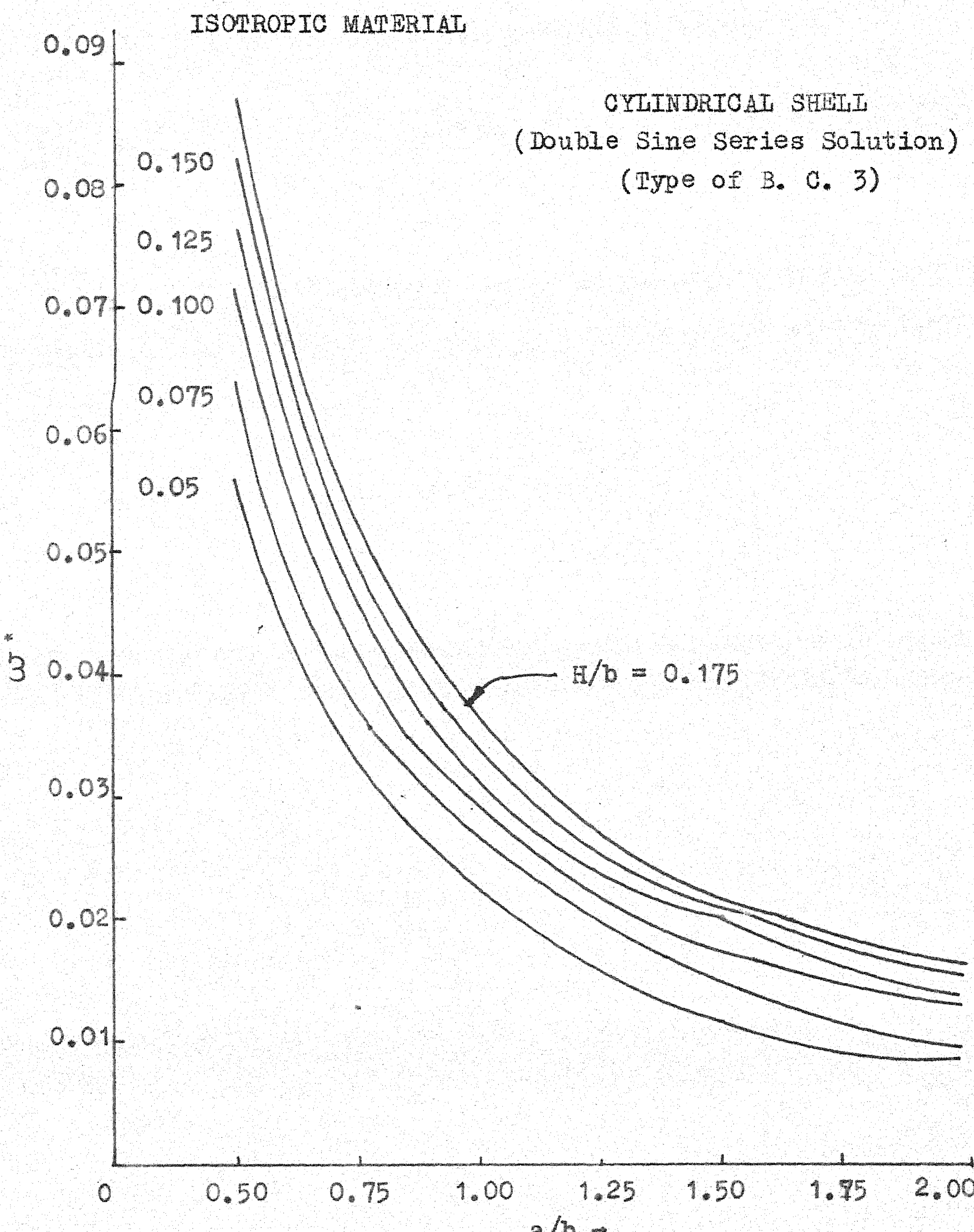
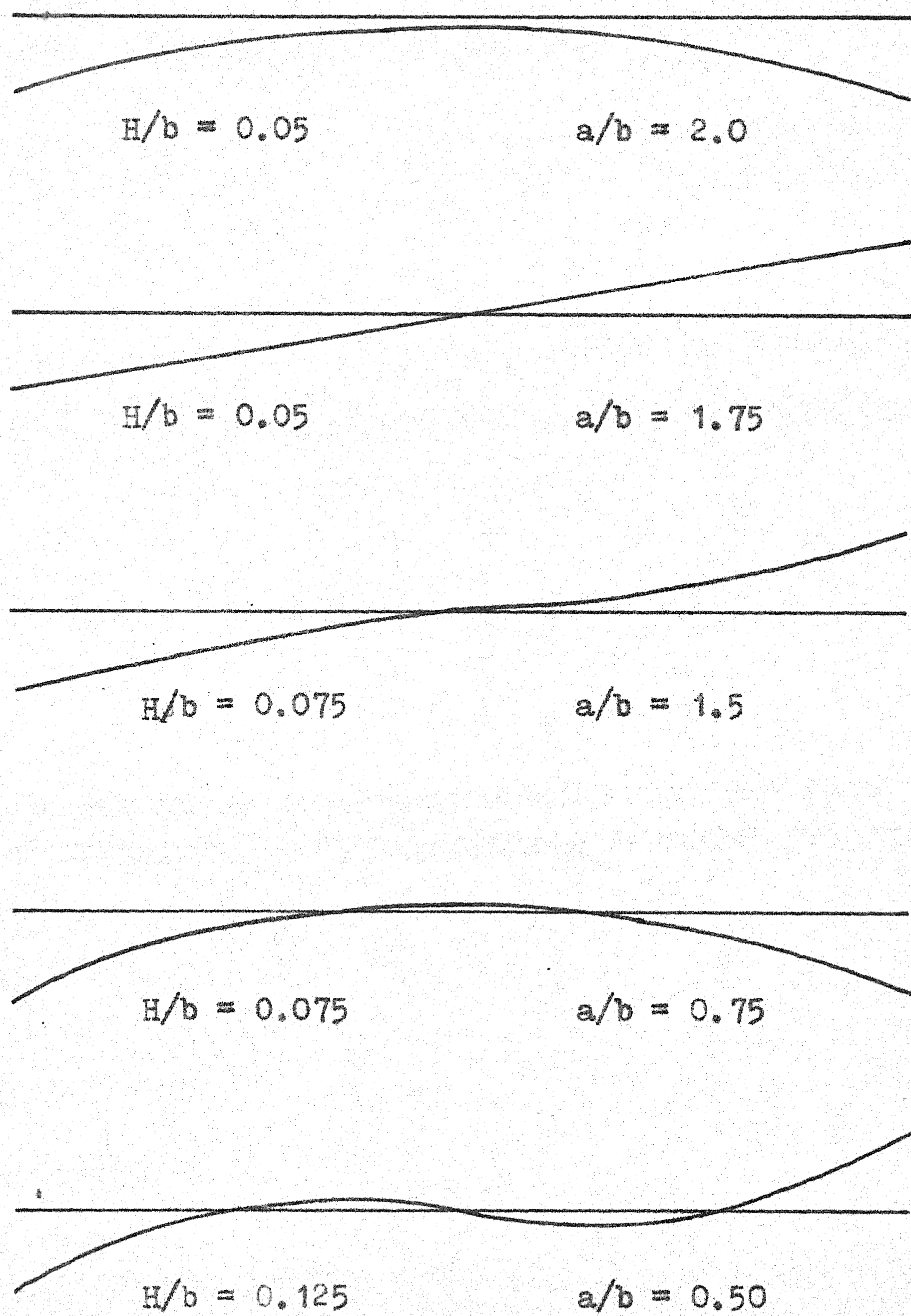

ISOTROPIC MATERIAL
CYLINDRICAL SHELL
(Double Sine Series Solution)
(Type of B. C. 3)
0.09
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0.150
0.125
0.100
0.075
0.05
H/b = 0.175
0
0.50
0.75
1.00
1.25
1.50
1.75
2.00
a/b →

Figure 5.11



TYPICAL EIGEN VECTORS
(For Type of B.C. 3)
Figure 5.12

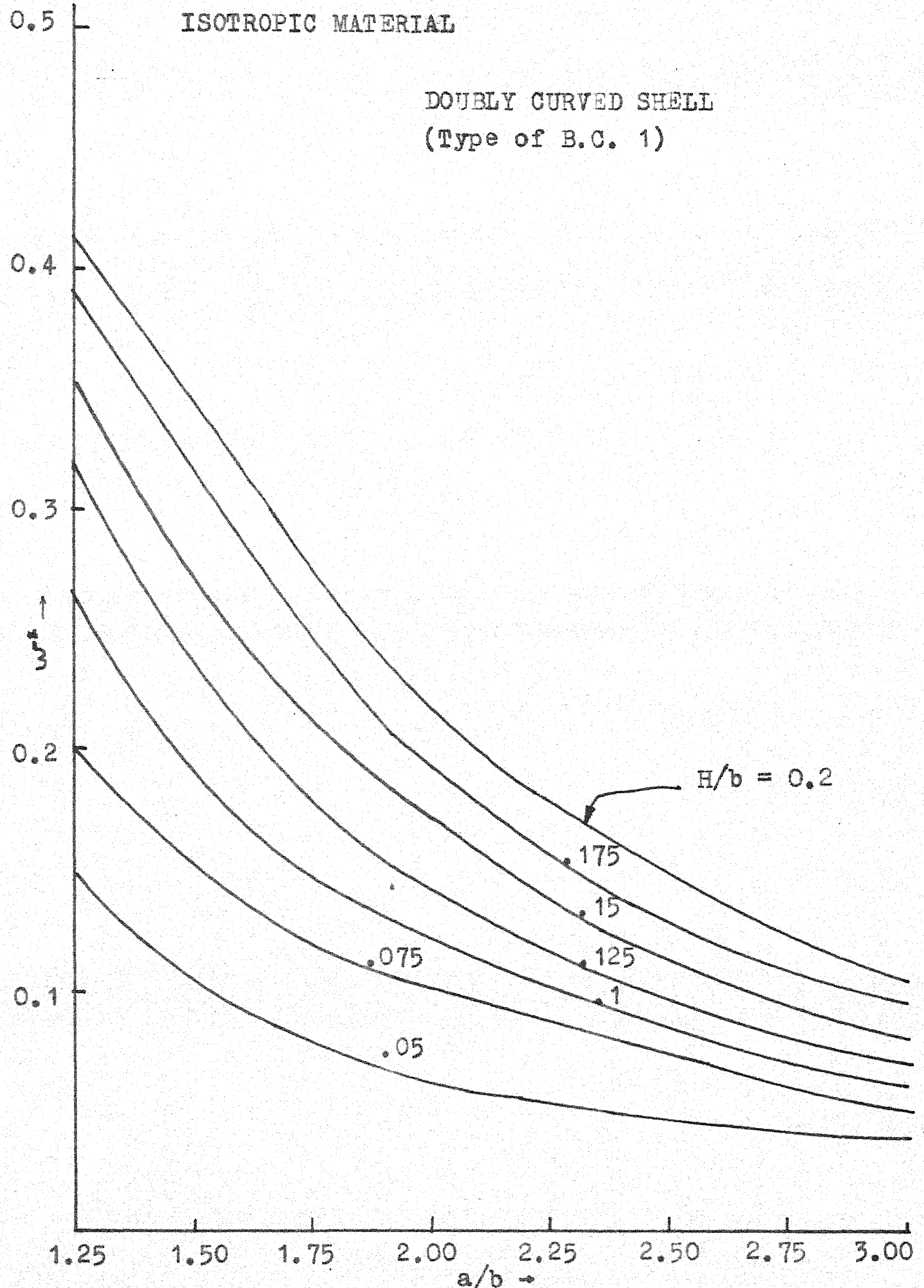

ISOTROPIC MATERIAL
DOUBLY CURVED SHELL
(Type of B.C. 1)
H/b = 0.2
.175
.15
.125
.1
.075
.05
0.5
0.4
0.3
0.2
0.1
ω^* ↑
1.25
1.50
1.75
2.00
2.25
2.50
2.75
3.00
a/b →

Figure 5.13

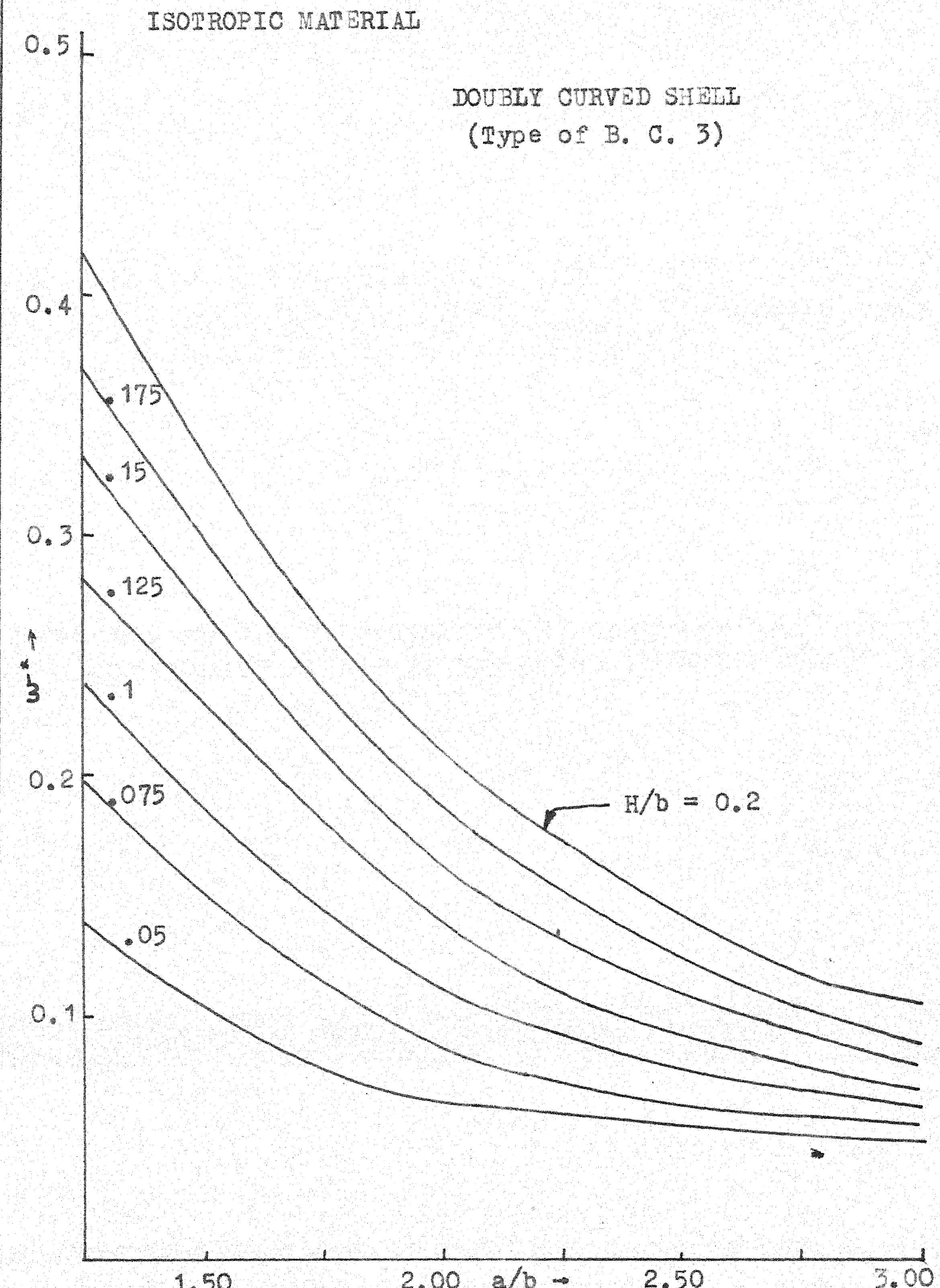

ISOTROPIC MATERIAL
DOUBLY CURVED SHELL
(Type of B. C. 3)
0.5
0.4
0.3
0.2
0.1
.175
.15
.125
.1
.075
.05
H/b = 0.2
1.50
2.00
a/b →
2.50
3.00

Figure 5.14

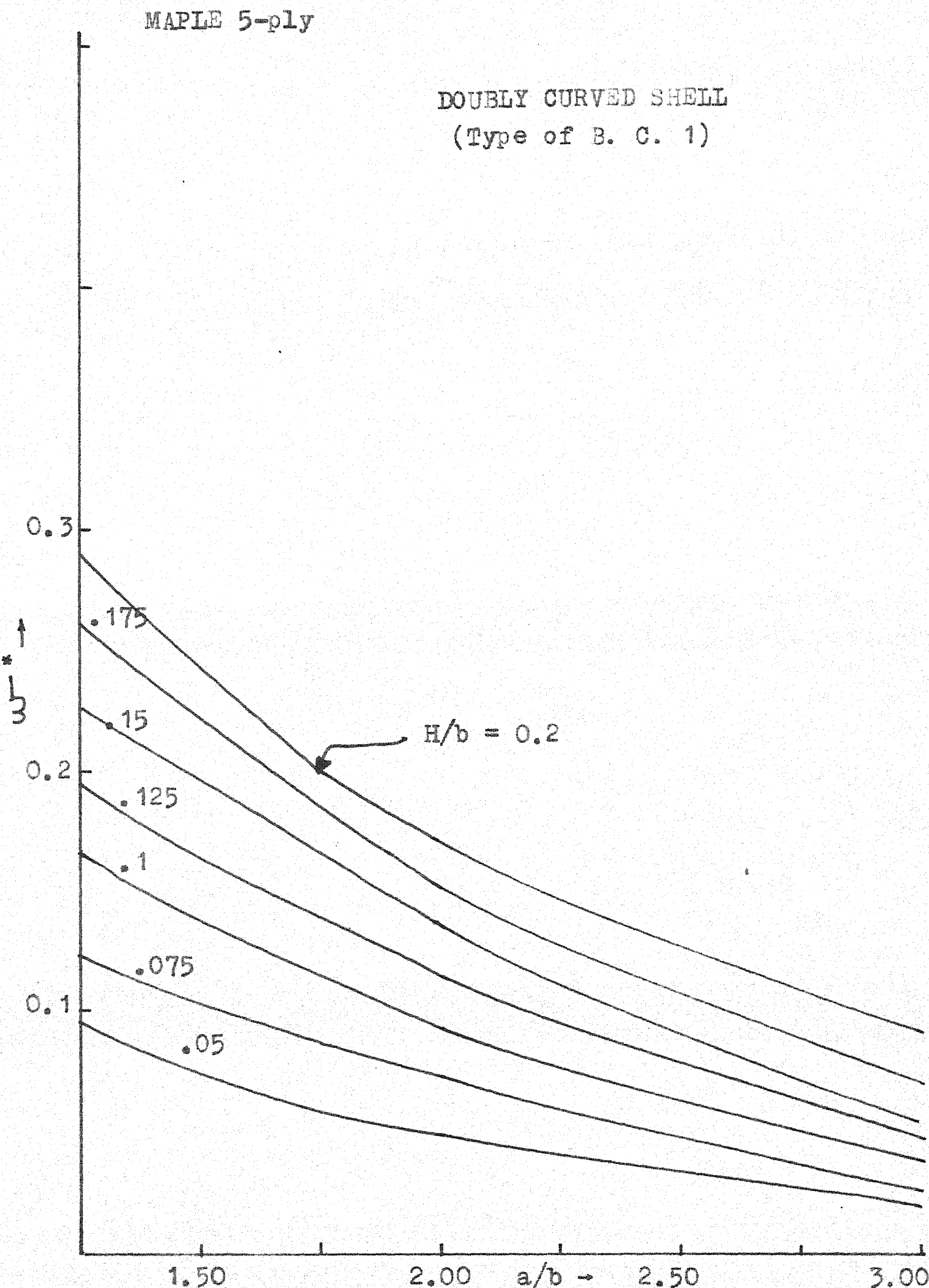

MAPLE 5-ply
DOUBLY CURVED SHELL
(Type of B. C. 1)
0.3
0.2
0.1
ω^* ↑
.175
.15
.125
.1
.075
.05
H/b = 0.2
1.50
2.00
a/b →
2.50
3.00

Figure 5.15

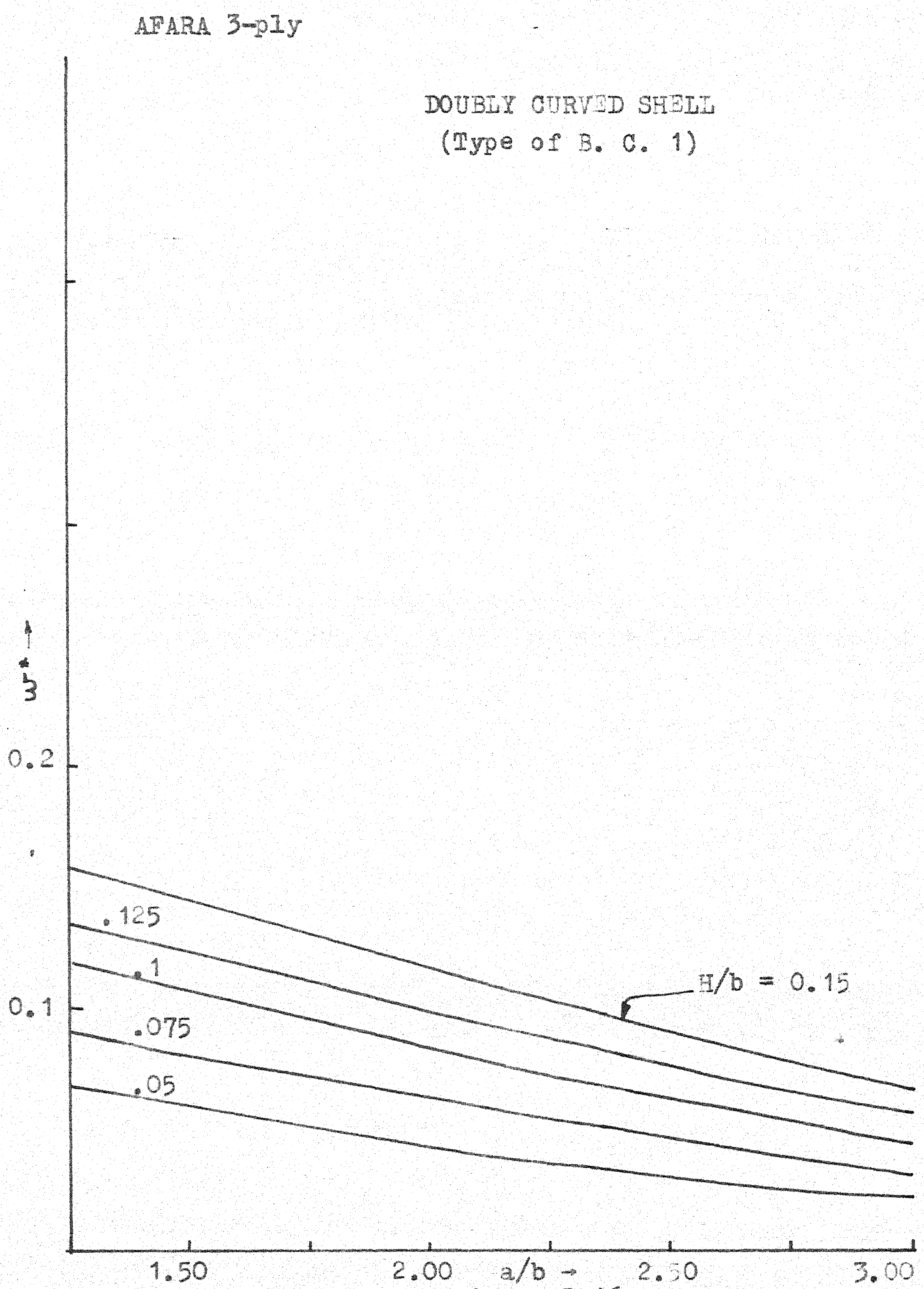

AFARA 3-ply
DOUBLY CURVED SHELL
(Type of B. C. 1)
$\bar{w}^*$ →
0.2
0.1
.125
.1
.075
.05
H/b = 0.15
1.50
2.00
a/b →
2.50
3.00

Figure 5.16

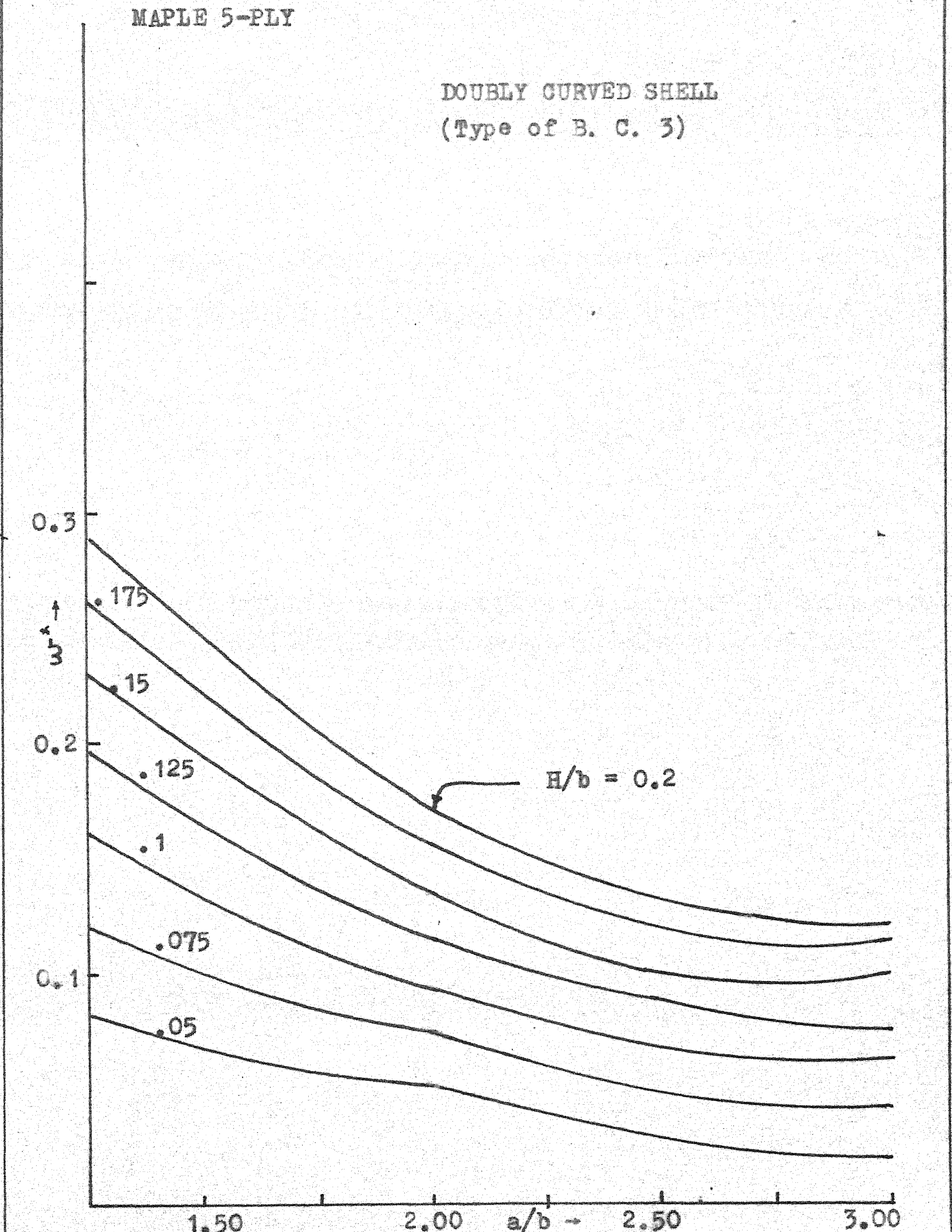

MAPLE 5-PLY
DOUBLY CURVED SHELL
(Type of B. C. 3)
0.3
0.2
0.1
.175
.15
.125
.1
.075
.05
H/b = 0.2
1.50
2.00
a/b →
2.50
3.00

Figure 5.17

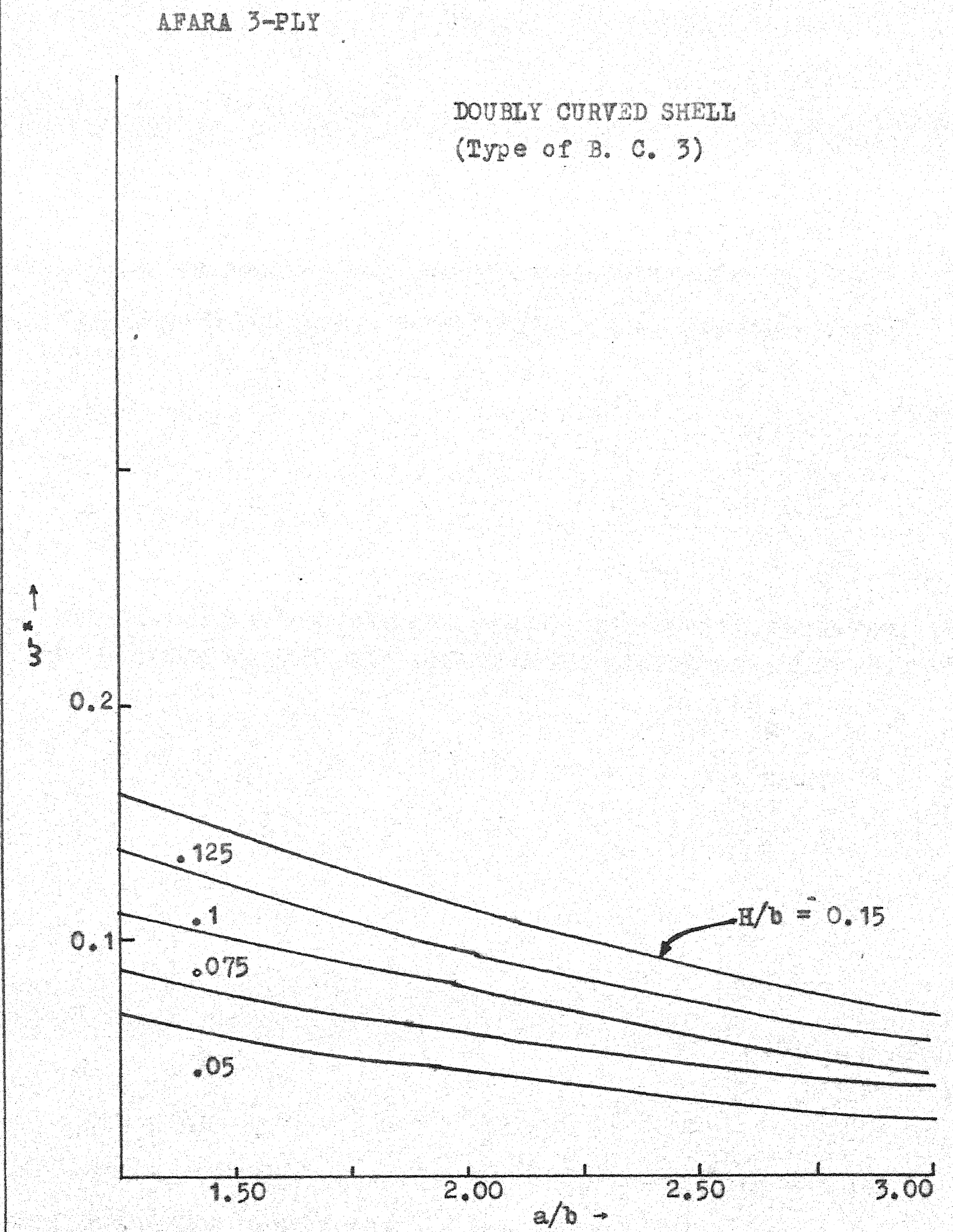

AFARA 3-PLY
DOUBLY CURVED SHELL
(Type of B. C. 3)
ω^* →
0.2
0.1
.125
.1
.075
.05
H/b = 0.15
1.50
2.00
2.50
3.00
a/b →

Figure 5.18

APPENDIX

SALIENT FEATURES OF COMPUTER PROGRAMME

The programme is written for calculating the natural frequencies and mode shapes for doubly curved anisotropic shells. All expressions are non-dimensionalised: lengths with characteristic length (b) Figure (5.1), Forces and shears with (B_{11} * b), moments with (B_{11} * b^2). Non-dimensional frequency ratio is defined as:

$$\omega = \omega^* \sqrt{\frac{\gamma^* b^2}{g B_{11}}} \tag{A-1}$$

The lengths α_o and β_o along the coordinate lines α and β respectively are taken as absolute lengths. This gives for shallow shells, the lame parameters as

$$A = B = 1 \tag{A-2}$$

The characteristic roots m_i are calculated and classified according to (4.4). $\{Y_R\}$ and $[N_R]$ are given by section (4.1) and the boundary conditions are developed according to sections (4.2) and (4.3) at initial and final boundary. The boundary conditions being homogeneous, these equations are homogeneous and can be written in matrix form as:

$$[EQN]\{C\} = 0 \tag{A-3}$$

where [EQN] is 8 x 8 matrix and C is 8 x 1 column vector of unknown variables. The existance of non-trivial solution of (A-3) ensures natural frequencies, and the lowest of them is the fundamental one. The condition for the same is that determinent of [EQN] whould vanish. It is minimised by changing the value of ω^*.

A graph of determinent verses ω^* is plotted. The frequency is given by ω^* when determinent changes sign or becomes zero. An iterative procedure such as the method of bisection is used to obtain accurate results.

FLOW CHART

START

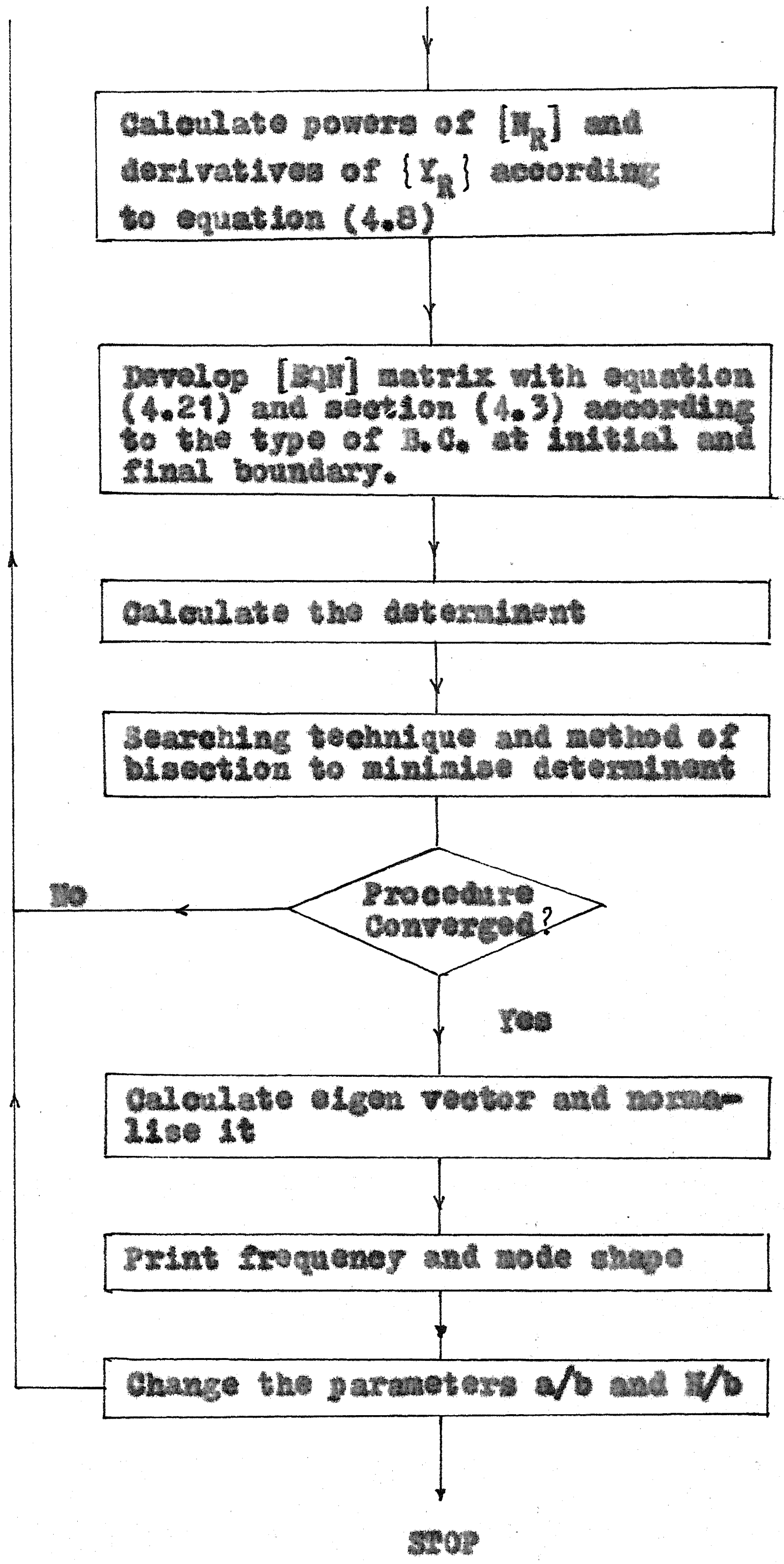
INPUT:

1. Title of the problem
2. Shell parameters: height (H), lengths (α_0, β_0), thickness (h), radii of curvatures (R_1, R_2), Lame parameters (A, B)
3. Material properties: B_{11}, B_{12}, B_{16}, B_{22}, B_{26}, B_{66}
4. Starting value of frequency ω^*, increment (p)

Calculate the coefficient of eighth order equation (4.3b)

Calculate characteristic roots m_i and classify them as in (4.4)

Develop the function $\{Y_R\}$ and $[N_R]$ according to sections (4.1) and (4.2) at initial and final boundaries


Calculate powers of $[N_R]$ and derivatives of $\{Y_R\}$ according to equation (4.8)
Develop [EQN] matrix with equation (4.21) and section (4.3) according to the type of B.C. at initial and final boundary.
Calculate the determinant
Searching technique and method of bisection to minimise determinant
Procedure Converged?
No
Yes
Calculate eigen vector and normalise it
Print frequency and mode shape
Change the parameters a/b and H/b
STOP

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